

# Thomas Calculus Early Transcendentals 14th Edition Hass **SOLUTIONS** **MANUAL**

## CHAPTER 2 LIMITS AND CONTINUITY

### 2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a)  $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{28-9}{1} = 19$  (b)  $\frac{\Delta f}{\Delta x} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$
2. (a)  $\frac{\Delta g}{\Delta x} = \frac{g(3)-g(1)}{3-1} = \frac{3-(-1)}{2} = 2$  (b)  $\frac{\Delta g}{\Delta x} = \frac{g(4)-g(-2)}{4-(-2)} = \frac{8-8}{6} = 0$
3. (a)  $\frac{\Delta h}{\Delta t} = \frac{h(\frac{3\pi}{4})-h(\frac{\pi}{4})}{\frac{3\pi}{4}-\frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$  (b)  $\frac{\Delta h}{\Delta t} = \frac{h(\frac{\pi}{6})-h(\frac{\pi}{3})}{\frac{\pi}{6}-\frac{\pi}{3}} = \frac{0-\sqrt{3}}{-\frac{\pi}{6}} = \frac{-3}{\pi}\sqrt{3}$
4. (a)  $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(0)}{\pi-0} = \frac{(2-1)-(2+1)}{\pi-0} = -\frac{2}{\pi}$  (b)  $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(-\pi)}{\pi-(-\pi)} = \frac{(2-1)-(2-1)}{2\pi} = 0$
5.  $\frac{\Delta R}{\Delta \theta} = \frac{R(2)-R(0)}{2-0} = \frac{\sqrt{8+1}-1}{2} = \frac{3-1}{2} = 1$
6.  $\frac{\Delta P}{\Delta \theta} = \frac{P(2)-P(1)}{2-1} = \frac{(8-16+10)-(1-4+5)}{1} = 2-2 = 0$
7. (a)  $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-5)-(2^2-5)}{h} = \frac{4+4h+h^2-5+1}{h} = \frac{4h+h^2}{h} = 4+h$ . As  $h \rightarrow 0$ ,  $4+h \rightarrow 4 \Rightarrow$  at  $P(2, -1)$  the slope is 4.  
(b)  $y - (-1) = 4(x - 2) \Rightarrow y + 1 = 4x - 8 \Rightarrow y = 4x - 9$
8. (a)  $\frac{\Delta y}{\Delta x} = \frac{(7-(2+h)^2)-(7-2^2)}{h} = \frac{7-4-4h-h^2-3}{h} = \frac{-4h-h^2}{h} = -4-h$ . As  $h \rightarrow 0$ ,  $-4-h \rightarrow -4 \Rightarrow$  at  $P(2, 3)$  the slope is -4.  
(b)  $y - 3 = (-4)(x - 2) \Rightarrow y - 3 = -4x + 8 \Rightarrow y = -4x + 11$
9. (a)  $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-2(2+h)-3)-(2^2-2(2)-3)}{h} = \frac{4+4h+h^2-4-2h-3-(-3)}{h} = \frac{2h+h^2}{h} = 2+h$ . As  $h \rightarrow 0$ ,  $2+h \rightarrow 2 \Rightarrow$  at  $P(2, -3)$  the slope is 2.  
(b)  $y - (-3) = 2(x - 2) \Rightarrow y + 3 = 2x - 4 \Rightarrow y = 2x - 7$ .
10. (a)  $\frac{\Delta y}{\Delta x} = \frac{((1+h)^2-4(1+h))-(1^2-4(1))}{h} = \frac{1+2h+h^2-4-4h-(-3)}{h} = \frac{h^2-2h}{h} = h-2$ . As  $h \rightarrow 0$ ,  $h-2 \rightarrow -2 \Rightarrow$  at  $P(1, -3)$  the slope is -2.  
(b)  $y - (-3) = (-2)(x - 1) \Rightarrow y + 3 = -2x + 2 \Rightarrow y = -2x - 1$ .

11. (a)  $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 2^3}{h} = \frac{8+12h+4h^2+h^3-8}{h} = \frac{12h+4h^2+h^3}{h} = 12+4h+h^2$ . As  $h \rightarrow 0$ ,  $12+4h+h^2 \rightarrow 12$ ,  $\Rightarrow$  at  $P(2, 8)$  the slope is 12.  
 (b)  $y - 8 = 12(x - 2) \Rightarrow y - 8 = 12x - 24 \Rightarrow y = 12x - 16$ .
12. (a)  $\frac{\Delta y}{\Delta x} = \frac{2-(1+h)^3-(2-1^3)}{h} = \frac{2-1-3h-3h^2-h^3-1}{h} = \frac{-3h-3h^2-h^3}{h} = -3-3h-h^2$ . As  $h \rightarrow 0$ ,  $-3-3h-h^2 \rightarrow -3$ ,  $\Rightarrow$  at  $P(1, 1)$  the slope is  $-3$ .  
 (b)  $y - 1 = (-3)(x - 1) \Rightarrow y - 1 = -3x + 3 \Rightarrow y = -3x + 4$ .

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$$13. (a) \frac{\Delta y}{\Delta x} = \frac{(1+h)^3 - 12(1+h) - (1^3 - 12(1))}{h} = \frac{1+3h+3h^2+h^3-12-12h-(-11)}{h} = \frac{-9h+3h^2+h^3}{h} = -9+3h+h^2.$$

As  $h \rightarrow 0$ ,  $-9+3h+h^2 \rightarrow -9 \Rightarrow$  at  $P(1, -11)$  the slope is  $-9$ .

$$(b) y - (-11) = (-9)(x - 1) \Rightarrow y + 11 = -9x + 9 \Rightarrow y = -9x - 2.$$

$$14. (a) \frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 3(2+h)^2 + 4 - (2^3 - 3(2)^2 + 4)}{h} = \frac{8+12h+6h^2+h^3-12-12h-3h^2+4-0}{h} = \frac{3h^2+h^3}{h} = 3h+h^2.$$

As  $h \rightarrow 0$ ,  $3h+h^2 \rightarrow 0 \Rightarrow$  at  $P(2, 0)$  the slope is  $0$ .

$$(b) y - 0 = 0(x - 2) \Rightarrow y = 0.$$

$$15. (a) \frac{\Delta y}{\Delta x} = \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h} = \frac{2+(-2+h)}{2(-2+h)} \cdot \frac{1}{h} = \frac{1}{2(-2+h)}.$$

As  $h \rightarrow 0$ ,  $\frac{1}{2(-2+h)} \rightarrow \frac{-1}{4}$ ,  $\Rightarrow$  at  $P(-2, \frac{1}{2})$  the slope is  $\frac{-1}{4}$ .

$$(b) y - \left(\frac{1}{2}\right) = \frac{-1}{4}(x - (-2)) \Rightarrow y + \frac{1}{2} = \frac{-1}{4}x - \frac{1}{2} \Rightarrow y = \frac{-1}{4}x - 1$$

$$16. (a) \frac{\Delta y}{\Delta x} = \frac{\frac{(4+h)}{2-(4+h)} - \frac{4}{-2-4}}{h} = \left(\frac{4+h}{-2-h} + \frac{2}{1}\right) \cdot \frac{1}{h} = \frac{4+h+2(-2-h)}{-2-h} \cdot \frac{1}{h} = \frac{-1}{-2-h} = \frac{1}{2+h}.$$

As  $h \rightarrow 0$ ,  $\frac{1}{2+h} \rightarrow \frac{1}{2}$ ,  $\Rightarrow$  at  $P(4, -2)$  the slope is  $\frac{1}{2}$ .

$$(b) y - (-2) = \frac{1}{2}(x - 4) \Rightarrow y + 2 = \frac{1}{2}x - 2 \Rightarrow y = \frac{1}{2}x - 4$$

$$17. (a) \frac{\Delta y}{\Delta x} = \frac{\sqrt{4+h} - \sqrt{4}}{h} = \frac{4+h-2}{\sqrt{h}} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$$

As  $h \rightarrow 0$ ,  $\frac{1}{\sqrt{4+h}+2} \rightarrow \frac{1}{\sqrt{4}+2} = \frac{1}{4}$ ,  $\Rightarrow$  at  $P(4, 2)$  the slope is  $\frac{1}{4}$ .

$$(b) y - 2 = \frac{1}{4}(x - 4) \Rightarrow y - 2 = \frac{1}{4}x - 1 \Rightarrow y = \frac{1}{4}x + 1$$

$$18. (a) \frac{\Delta y}{\Delta x} = \frac{\sqrt{7-(-2+h)} - \sqrt{7-(-2)}}{h} = \frac{9-h-3}{\sqrt{h}} = \frac{9-h-3}{\sqrt{h}} \cdot \frac{\sqrt{9-h}+3}{\sqrt{9-h}+3} = \frac{(9-h)-9}{h(\sqrt{9-h}+3)} = \frac{-1}{\sqrt{9-h}+3}.$$

As  $h \rightarrow 0$ ,  $\frac{-1}{\sqrt{9-h}+3} \rightarrow \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$ ,  $\Rightarrow$  at  $P(-2, 3)$  the slope is  $\frac{-1}{6}$ .

$$(b) y - 3 = \frac{-1}{6}(x - (-2)) \Rightarrow y - 3 = \frac{-1}{6}x - \frac{1}{3} \Rightarrow y = \frac{-1}{6}x + \frac{8}{3}$$

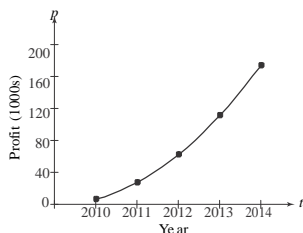
(a)	$Q$	Slope of $PQ = \frac{\Delta p}{\Delta t}$
	$Q_1(10, 225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
	$Q_2(14, 375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
	$Q_3(16.5, 475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
	$Q_4(18, 550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

(b) At  $t = 20$ , the sportscar was traveling approximately 50 m/sec or 180 km/h.

(a)	$Q$	Slope of $PQ = \frac{\Delta p}{\Delta t}$
	$Q_1(5, 20)$	$\frac{80-20}{10-5} = 12$ m/sec
	$Q_2(7, 39)$	$\frac{80-39}{10-7} = 13.7$ m/sec
	$Q_3(8.5, 58)$	$\frac{80-58}{10-8.5} = 14.7$ m/sec
	$Q_4(9.5, 72)$	$\frac{80-72}{10-9.5} = 16$ m/sec

(b) Approximately 16 m/sec

21. (a)



(b)  $\frac{\Delta p}{\Delta t} = \frac{174-62}{2014-2012} = \frac{112}{2} = 56$  thousand dollars per year

(c) The average rate of change from 2011 to 2012 is  $\frac{\Delta p}{\Delta t} = \frac{62-27}{2012-2011} = 35$  thousand dollars per year.

The average rate of change from 2012 to 2013 is  $\frac{\Delta p}{\Delta t} = \frac{111-62}{2013-2012} = 49$  thousand dollars per year.

 So, the rate at which profits were changing in 2012 is approximately  $\frac{1}{2}(35 + 49) = 42$  thousand dollars per year.

 22. (a)  $F(x) = (x+2)/(x-2)$ 

$x$	1.2	1.1	1.01	1.001	1.0001	1
$F(x)$	-4.0	-3.4	-3.04	-3.004	-3.0004	-3

$$\frac{\Delta F}{\Delta x} = \frac{-4.0-(-3)}{1.2-1} = -5.0;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.4-(-3)}{1.1-1} = -4.4;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.04-(-3)}{1.01-1} = -4.04;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.004-(-3)}{1.001-1} = -4.004;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.0004-(-3)}{1.0001-1} = -4.0004;$$

 (b) The rate of change of  $F(x)$  at  $x = 1$  is  $-4$ .

23. (a)  $\frac{\Delta g}{\Delta x} = \frac{g(2)-g(1)}{2-1} = \frac{2-1}{\sqrt{2}-1} \approx 0.414213$

$$\frac{\Delta g}{\Delta x} = \frac{g(1.5)-g(1)}{1.5-1} = \frac{1.5-1}{\sqrt{0.5}} \approx 0.449489$$

$$\frac{\Delta g}{\Delta x} = \frac{g(1+h)-g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$$

 (b)  $g(x) = \sqrt{x}$ 

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$(\sqrt{1+h}-1)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

 (c) The rate of change of  $g(x)$  at  $x = 1$  is  $0.5$ .

 (d) The calculator gives  $\lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$ .

24. (a) i)  $\frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{-\frac{1}{6}}{1} = -\frac{1}{6}$

ii)  $\frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2-T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = -\frac{1}{2T}, T \neq 2$

$T$	2.1	2.01	2.001	2.0001	2.00001	2.000001
$f(T)$	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
$(f(T)-f(2))/(T-2)$	-0.2381	-0.2488	-0.2500	-0.2500	-0.2500	-0.2500

 (c) The table indicates the rate of change is  $-0.25$  at  $t = 2$ .

(d)  $\lim_{T \rightarrow 2} \left( \frac{1}{-2T} \right) = -\frac{1}{4}$

NOTE: Answers will vary in Exercises 25 and 26.

25. (a)  $[0, 1]: \frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$  mph;  $[1, 2.5]: \frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$  mph;  $[2.5, 3.5]: \frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$  mph

- (b) At  $P(\frac{1}{2}, 7.5)$ : Since the portion of the graph from  $t = 0$  to  $t = 1$  is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at  $t = \frac{1}{2}$  is  $\frac{15-7.5}{1-0.5} = 15$  mi/hr. At  $P(2, 20)$ : Since the portion of the graph from  $t = 2$  to  $t = 2.5$  is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus  $v = \frac{20-20}{2.5-2} = 0$  mi/hr. For values of  $t$  less than 2, we have

$Q$	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at  $t = 2$  is 0 mi/hr.

At  $P(3, 22)$ :

$Q$	Slope of $PQ = \frac{\Delta s}{\Delta t}$	$Q$	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr	$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr	$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr	$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at  $t = 3$  is about 7 mi/hr.

- (c) It appears that the curve is increasing the fastest at  $t = 3.5$ . Thus for  $P(3.5, 30)$

$Q$	Slope of $PQ = \frac{\Delta s}{\Delta t}$	$Q$	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr	$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr	$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr	$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at  $t = 3.5$  is about 20 mi/hr.

26. (a)  $[0, 3]: \frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67 \frac{\text{gal}}{\text{day}}$ ;  $[0, 5]: \frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2 \frac{\text{gal}}{\text{day}}$ ;  $[7, 10]: \frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5 \frac{\text{gal}}{\text{day}}$

- (b) At  $P(1, 14)$ :

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$	$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day	$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day	$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day	$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at  $t = 1$  is about  $-1.45$  gal/day.

At  $P(4, 6)$ :

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$	$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day	$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day	$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day	$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at  $t = 4$  is  $-3$  gal/day.

(solution continues on next page)

At  $P(8, 1)$ :

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at  $t = 1$  is  $-0.55$  gal/day.

- (c) It appears that the curve (the consumption) is decreasing the fastest at  $t = 3.5$ . Thus for  $P(3.5, 7.8)$

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

$Q$	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at  $t = 3.5$  is about  $-4$  gal/day.

## 2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

- Does not exist. As  $x$  approaches 1 from the right,  $g(x)$  approaches 0. As  $x$  approaches 1 from the left,  $g(x)$  approaches 1. There is no single number  $L$  that all the values  $g(x)$  get arbitrarily close to as  $x \rightarrow 1$ .
  - 1
  - 0
  - 0.5
- 0
  - 1
  - Does not exist. As  $t$  approaches 0 from the left,  $f(t)$  approaches -1. As  $t$  approaches 0 from the right,  $f(t)$  approaches 1. There is no single number  $L$  that  $f(t)$  gets arbitrarily close to as  $t \rightarrow 0$ .
  - 1
- True
  - False
  - True
  - True
  - True
  - True
  - False
  - False
  - False
  - False
- False
  - True
  - False
  - False
  - True
  - True
  - True
  - True
  - True
  - False
- $\lim_{x \rightarrow 0} \frac{x}{|x|}$  does not exist because  $\frac{x}{|x|} = \frac{x}{x} = 1$  if  $x > 0$  and  $\frac{x}{|x|} = \frac{x}{-x} = -1$  if  $x < 0$ . As  $x$  approaches 0 from the left,  $\frac{x}{|x|}$  approaches -1. As  $x$  approaches 0 from the right,  $\frac{x}{|x|}$  approaches 1. There is no single number  $L$  that all the function values get arbitrarily close to as  $x \rightarrow 0$ .
- As  $x$  approaches 1 from the left, the values of  $\frac{1}{x-1}$  become increasingly large and negative. As  $x$  approaches 1 from the right, the values become increasingly large and positive. There is no number  $L$  that all the function values get arbitrarily close to as  $x \rightarrow 1$ , so  $\lim_{x \rightarrow 1} \frac{1}{x-1}$  does not exist.
- Nothing can be said about  $f(x)$  because the existence of a limit as  $x \rightarrow x_0$  does not depend on how the function is defined at  $x_0$ . In order for a limit to exist,  $f(x)$  must be arbitrarily close to a single real number  $L$  when  $x$  is

close enough to  $x_0$ . That is, the existence of a limit depends on the values of  $f(x)$  for  $x$  near  $x_0$ , not on the definition of  $f(x)$  at  $x_0$  itself.

8. Nothing can be said. In order for  $\lim_{x \rightarrow 0} f(x)$  to exist,  $f(x)$  must close to a single value for  $x$  near 0 regardless of the value  $f(0)$  itself.
9. No, the definition does not require that  $f$  be defined at  $x = 1$  in order for a limiting value to exist there. If  $f(1)$  is defined, it can be any real number, so we can conclude nothing about  $f(1)$  from  $\lim_{x \rightarrow 1} f(x) = 5$ .
10. No, because the existence of a limit depends on the values of  $f(x)$  when  $x$  is near 1, not on  $f(1)$  itself. If  $\lim_{x \rightarrow 1} f(x)$  exists, its value may be some number other than  $f(1) = 5$ . We can conclude nothing about  $\lim_{x \rightarrow 1} f(x)$ , whether it exists or what its value is if it does exist, from knowing the value of  $f(1)$  alone.
11.  $\lim_{x \rightarrow -3} (x^2 - 13) = (-3)^2 - 13 = 9 - 13 = -4$
12.  $\lim_{x \rightarrow 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$
13.  $\lim_{t \rightarrow 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$
14.  $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$
15.  $\lim_{x \rightarrow 2} \frac{2x+5}{11-x^3} = \frac{2(2)+5}{11-(2)^3} = \frac{9}{3} = 3$
16.  $\lim_{t \rightarrow 2/3} (8 - 3s)(2s - 1) = \left(8 - 5\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right) - 1\right) = (8 - 2)\left(\left(\frac{4}{3}\right) - 1\right) = (6)\left(\frac{1}{3}\right) = 2$
17.  $\lim_{x \rightarrow -1/2} 4x(3x + 4)^2 = 4\left(-\frac{1}{2}\right)\left(3\left(-\frac{1}{2}\right) + 4\right)^2 = (-2)\left(-\frac{3}{2} + 4\right)^2 = (-2)\left(\frac{5}{2}\right)^2 = -\frac{25}{2}$
18.  $\lim_{y \rightarrow 2} \frac{y+2}{y^2+5y+6} = \frac{2+2}{(2)^2+5(2)+6} = \frac{4}{4+10+6} = \frac{4}{20} = \frac{1}{5}$
19.  $\lim_{y \rightarrow -3} (5 - y)^{4/3} = [5 - (-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$
20.  $\lim_{z \rightarrow 4} \sqrt{z^2 - 10} = \sqrt{4^2 - 10} = \frac{16 - 10}{\sqrt{\quad}} = \frac{6}{\sqrt{\quad}}$
21.  $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{3(0)+1+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$
22.  $\lim_{h \rightarrow 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \lim_{h \rightarrow 0} \frac{(5h+4)-4}{h(\sqrt{5h+4}+2)} = \lim_{h \rightarrow 0} \frac{5h}{h(\sqrt{5h+4}+2)} = \lim_{h \rightarrow 0} \frac{5}{\sqrt{5h+4}+2} = \frac{5}{\sqrt{4}+2} = \frac{5}{4}$
23.  $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \rightarrow 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$
24.  $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3} = \lim_{x \rightarrow -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x \rightarrow -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$



$$25. \lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5} = \lim_{x \rightarrow -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \rightarrow -5} (x-2) = -5-2 = -7$$

$$26. \lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x \rightarrow 2} (x-5) = 2-5 = -3$$

$$27. \lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1} = \lim_{t \rightarrow 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \rightarrow 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$$

$$28. \lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2} = \lim_{t \rightarrow -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t \rightarrow -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$$

$$29. \lim_{x \rightarrow -2} \frac{-2x-4}{x^3 + 2x^2} = \lim_{x \rightarrow -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \rightarrow -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$30. \lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2} = \lim_{y \rightarrow 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \rightarrow 0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$$

$$31. \lim_{x \rightarrow 1} \frac{x^{-1}-1}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x \rightarrow 1} \left( \frac{1-x}{x} \cdot \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} -\frac{1}{x} = -1$$

$$32. \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} - \frac{1}{x+1}}{x} = \lim_{x \rightarrow 0} \frac{\frac{(x+1) + (x-1)}{(x-1)(x+1)}}{x} = \lim_{x \rightarrow 0} \left( \frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$$

$$33. \lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} = \lim_{u \rightarrow 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \rightarrow 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$$

$$34. \lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16} = \lim_{v \rightarrow 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v \rightarrow 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$$

$$35. \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$$36. \lim_{x \rightarrow 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x \rightarrow 4} x(2+\sqrt{x}) = 4(2+2) = 16$$

$$37. \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x \rightarrow 1} (\sqrt{x+3}+2) = \sqrt{4}+2 = 4$$

$$38. \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \rightarrow -1} \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \rightarrow -1} \frac{(x^2+8)-9}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \rightarrow -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} \\ = \lim_{x \rightarrow -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$$

$$39. \lim_{x \rightarrow 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x \rightarrow 2} \frac{(\sqrt{x^2+12}-4)(\sqrt{x^2+12}+4)}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x \rightarrow 2} \frac{(x^2+12)-16}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+12}+4)} \\ = \lim_{x \rightarrow 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$$

$$\lim_{x \rightarrow 2} \sqrt{x^2 + 12} + 4 = \sqrt{16 + 4} + 4 = 2 + 4 = 6$$

$$40. \lim_{x \rightarrow -2} \frac{x+2}{\sqrt{x^2+5}-3} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(x^2+5)-9} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(x+2)(x-2)}$$

$$= \lim_{x \rightarrow -2} \frac{\sqrt{x^2+5}+3}{x-2} = \frac{\sqrt{9+5}+3}{-4} = -\frac{3}{2}$$

$$41. \lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim_{x \rightarrow -3} \frac{(2-\sqrt{x^2-5})(2+\sqrt{x^2-5})}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{4-(x^2-5)}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{9-x^2}{(x+3)(2+\sqrt{x^2-5})}$$

$$= \lim_{x \rightarrow -3} \frac{(3-x)(3+x)}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{6}{2+\sqrt{4}} = \frac{3}{2}$$

$$42. \lim_{x \rightarrow 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(5-\sqrt{x^2+9})(5+\sqrt{x^2+9})} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{25-(x^2+9)} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2}$$

$$= \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(4-x)(4+x)} = \lim_{x \rightarrow 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5+\sqrt{25}}{8} = \frac{5}{4}$$

$$43. \lim_{x \rightarrow 0} (2 \sin x - 1) = 2 \sin 0 - 1 = 0 - 1 = -1$$

$$44. \lim_{x \rightarrow 0} \sin^2 x = \left( \lim_{x \rightarrow 0} \sin x \right)^2 = (\sin 0)^2 = 0^2 = 0$$

$$45. \lim_{x \rightarrow 0} \sec x = \lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$46. \lim_{x \rightarrow 0} \tan x = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$47. \lim_{x \rightarrow 0} \frac{1+x+\sin x}{3 \cos x} = \frac{1+0+\sin 0}{3 \cos 0} = \frac{1+0+0}{3} = \frac{1}{3}$$

$$48. \lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x) = (0^2 - 1)(2 - \cos 0) = (-1)(2 - 1) = (-1)(1) = -1$$

$$49. \lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi) = \lim_{x \rightarrow -\pi} \sqrt{x+4} \cdot \lim_{x \rightarrow -\pi} \cos(x+\pi) = \sqrt{-\pi+4} \cdot \cos 0 = \sqrt{4-\pi} \cdot 1 = \sqrt{4-\pi}$$

$$50. \lim_{x \rightarrow 0} \sqrt{7 + \sec^2 x} = \sqrt{\lim_{x \rightarrow 0} (7 + \sec^2 x)} = \sqrt{7 + \lim_{x \rightarrow 0} \sec^2 x} = \sqrt{7 + \sec^2 0} = \sqrt{7 + (1)^2} = 2\sqrt{2}$$

51. (a) quotient rule (b) difference and power rules  
(c) sum and constant multiple rules

52. (a) quotient rule (b) power and product rules  
(c) difference and constant multiple rules

$$53. (a) \lim_{x \rightarrow c} f(x)g(x) = \left[ \lim_{x \rightarrow c} f(x) \right] \left[ \lim_{x \rightarrow c} g(x) \right] = (5)(-2) = -10$$

$$(b) \lim_{x \rightarrow c} 2f(x)g(x) = 2 \left[ \lim_{x \rightarrow c} f(x) \right] \left[ \lim_{x \rightarrow c} g(x) \right] = 2(5)(-2) = -20$$

$$(c) \lim_{x \rightarrow c} [f(x) + 3g(x)] = \lim_{x \rightarrow c} f(x) + 3 \lim_{x \rightarrow c} g(x) = 5 + 3(-2) = -1$$

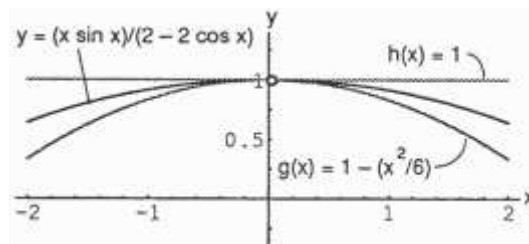
$$\frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{5}{-2} = -\frac{5}{2}$$

$$(d) \lim_{x \rightarrow c} f(x) - g(x) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = 5 - (-2) = 7$$

54. (a)  $\lim_{x \rightarrow 4} [g(x) + 3] = \lim_{x \rightarrow 4} g(x) + \lim_{x \rightarrow 4} 3 = -3 + 3 = 0$   
 (b)  $\lim_{x \rightarrow 4} xf(x) = \lim_{x \rightarrow 4} x \cdot \lim_{x \rightarrow 4} f(x) = (4)(0) = 0$   
 (c)  $\lim_{x \rightarrow 4} [g(x)]^2 = \left[ \lim_{x \rightarrow 4} g(x) \right]^2 = [-3]^2 = 9$   
 (d)  $\lim_{x \rightarrow 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x \rightarrow 4} g(x)}{\lim_{x \rightarrow 4} f(x) - \lim_{x \rightarrow 4} 1} = \frac{-3}{0-1} = 3$
55. (a)  $\lim_{x \rightarrow b} [f(x) + g(x)] = \lim_{x \rightarrow b} f(x) + \lim_{x \rightarrow b} g(x) = 7 + (-3) = 4$   
 (b)  $\lim_{x \rightarrow b} f(x) \cdot g(x) = \left[ \lim_{x \rightarrow b} f(x) \right] \left[ \lim_{x \rightarrow b} g(x) \right] = (7)(-3) = -21$   
 (c)  $\lim_{x \rightarrow b} 4g(x) = \left[ \lim_{x \rightarrow b} 4 \right] \left[ \lim_{x \rightarrow b} g(x) \right] = (4)(-3) = -12$   
 (d)  $\lim_{x \rightarrow b} f(x)/g(x) = \lim_{x \rightarrow b} f(x) / \lim_{x \rightarrow b} g(x) = \frac{7}{-3} = -\frac{7}{3}$
56. (a)  $\lim_{x \rightarrow -2} [p(x) + r(x) + s(x)] = \lim_{x \rightarrow -2} p(x) + \lim_{x \rightarrow -2} r(x) + \lim_{x \rightarrow -2} s(x) = 4 + 0 + (-3) = 1$   
 (b)  $\lim_{x \rightarrow -2} p(x) \cdot r(x) \cdot s(x) = \left[ \lim_{x \rightarrow -2} p(x) \right] \left[ \lim_{x \rightarrow -2} r(x) \right] \left[ \lim_{x \rightarrow -2} s(x) \right] = (4)(0)(-3) = 0$   
 (c)  $\lim_{x \rightarrow -2} [-4p(x) + 5r(x)]/s(x) = \frac{-4 \lim_{x \rightarrow -2} p(x) + 5 \lim_{x \rightarrow -2} r(x)}{\lim_{x \rightarrow -2} s(x)} = \frac{-4(4) + 5(0)}{-3} = \frac{16}{3}$
57.  $\lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow 0} \frac{1+2h+h^2-1}{h} = \lim_{h \rightarrow 0} \frac{h(2+h)}{h} = \lim_{h \rightarrow 0} (2+h) = 2$
58.  $\lim_{h \rightarrow 0} \frac{(-2+h)^2 - (-2)^2}{h} = \lim_{h \rightarrow 0} \frac{4-4h+h^2-4}{h} = \lim_{h \rightarrow 0} \frac{h(h-4)}{h} = \lim_{h \rightarrow 0} (h-4) = -4$
59.  $\lim_{h \rightarrow 0} \frac{[3(2+h)-4] - [3(2)-4]}{h} = \lim_{h \rightarrow 0} \frac{3h}{h} = 3$
60.  $\lim_{h \rightarrow 0} \frac{(-2+h)^2 - (-2)^2}{h} = \lim_{h \rightarrow 0} \frac{-2+h}{-2h} = \lim_{h \rightarrow 0} \frac{-2-(-2+h)}{-2h(-2+h)} = \lim_{h \rightarrow 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$
61.  $\lim_{h \rightarrow 0} \frac{\sqrt{7+h} - \sqrt{7}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{7+h} - \sqrt{7})(\sqrt{7+h} + \sqrt{7})}{h(\sqrt{7+h} + \sqrt{7})} = \lim_{h \rightarrow 0} \frac{(7+h)-7}{h(\sqrt{7+h} + \sqrt{7})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{7+h} + \sqrt{7})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{7+h} + \sqrt{7}} = \frac{1}{2\sqrt{7}}$
62.  $\lim_{h \rightarrow 0} \frac{\sqrt{3(0+h)+1} - \sqrt{3(0)+1}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{3h+1} - 2)(\sqrt{3h+1} + 2)}{h(\sqrt{3h+1} + 2)} = \lim_{h \rightarrow 0} \frac{(3h+1)-4}{h(\sqrt{3h+1} + 2)} = \lim_{h \rightarrow 0} \frac{3h-3}{h(\sqrt{3h+1} + 2)} = \lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1} + 2} = \frac{3}{4}$
63.  $\lim_{x \rightarrow 0} \sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$  and  $\lim_{x \rightarrow 0} \sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5}$ ; by the sandwich theorem,  $\lim_{x \rightarrow 0} f(x) = \sqrt{5}$
64.  $\lim_{x \rightarrow 0} (2-x^2) = 2-0 = 2$  and  $\lim_{x \rightarrow 0} 2 \cos x = 2(1) = 2$ ; by the sandwich theorem,  $\lim_{x \rightarrow 0} g(x) = 2$
65. (a)  $\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1$  and  $\lim_{x \rightarrow 0} 1 = 1$ ; by the sandwich theorem,  $\lim_{x \rightarrow 0} \frac{x \sin x}{2-2 \cos x} = 1$

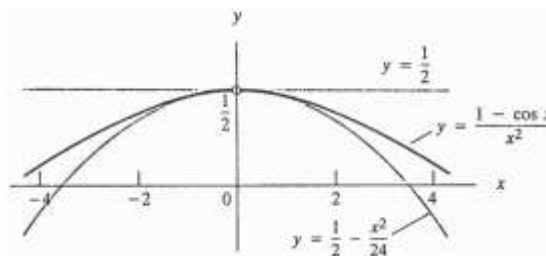
(b)

- (b) For  $x \neq 0$ ,  $y = (x \sin x)/(2 - 2 \cos x)$  lies between the other two graphs in the figure, and the graphs converge as  $x \rightarrow 0$ .



66. (a)  $\lim_{x \rightarrow 0} \left( \frac{1}{2} - \frac{x^2}{24} \right) = \lim_{x \rightarrow 0} \frac{1}{2} - \lim_{x \rightarrow 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$  and  $\lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$ ; by the sandwich theorem,  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ .

- (b) For all  $x \neq 0$ , the graph of  $f(x) = (1 - \cos x)/x^2$  lies between the line  $y = \frac{1}{2}$  and the parabola  $y = \frac{1}{2} - x^2/24$ , and the graphs converge as  $x \rightarrow 0$ .



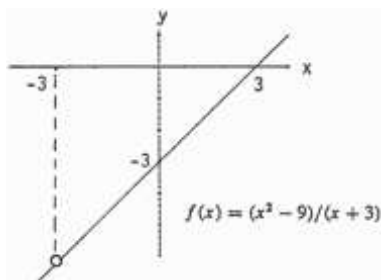
67. (a)  $f(x) = (x^2 - 9)/(x + 3)$

$x$	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

$x$	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

The estimate is  $\lim_{x \rightarrow -3} f(x) = -6$ .

(b)

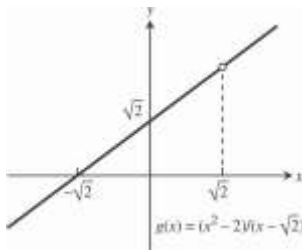


- (c)  $f(x) = \frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = x - 3$  if  $x \neq -3$ , and  $\lim_{x \rightarrow -3} (x - 3) = -3 - 3 = -6$ .

68. (a)  $g(x) = (x^2 - 2)/(x - \sqrt{2})$

$x$	1.4	1.41	1.414	1.4142	1.41421	1.414213
$g(x)$	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



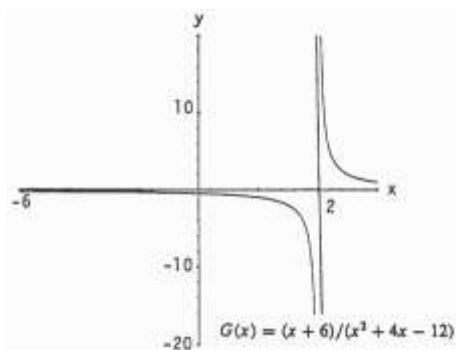
$$(c) \quad g(x) = \frac{x^2 - 2}{x - \sqrt{2}} = \frac{(x + \sqrt{2})(x - \sqrt{2})}{(x - \sqrt{2})} = x + \sqrt{2} \text{ if } x \neq \sqrt{2}, \text{ and } \lim_{x \rightarrow \sqrt{2}} (x + \sqrt{2}) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

69. (a)  $G(x) = (x+6)/(x^2 + 4x - 12)$

$x$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999
$G(x)$	-.126582	-.1251564	-.1250156	-.1250015	-.1250001	-.1250000

$x$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001
$G(x)$	-.123456	-.124843	-.124984	-.124998	-.124999	-.124999

(b)



$$(c) \quad G(x) = \frac{x+6}{x^2 + 4x - 12} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2} \text{ if } x \neq -6, \text{ and } \lim_{x \rightarrow -6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125.$$

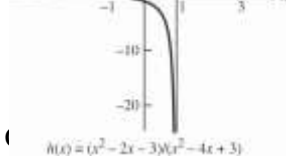
70. (a)  $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$

$x$	2.9	2.99	2.999	2.9999	2.99999	2.999999
$h(x)$	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

$x$	3.1	3.01	3.001	3.0001	3.00001	3.000001
$h(x)$	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)

$$(c) \quad h(x) = \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1} \text{ if } x \neq 3, \text{ and } \lim_{x \rightarrow 3} \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2.$$



$$\lim_{x \rightarrow 3^-} (b) \quad \lim_{x \rightarrow 3^-} = 2 =$$

$$x \rightarrow$$

(b)

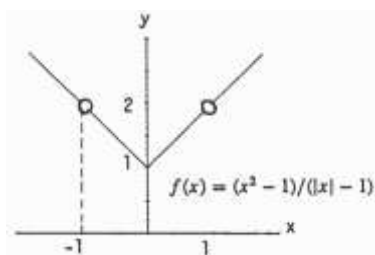
71. (a)  $f(x) = (x^2 - 1)/(|x| - 1)$

$x$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

$x$	-0.9	-0.99	-0.999	-0.9999	-0.99999	-0.999999
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



(c)  $f(x) = \frac{x^2 - 1}{|x| - 1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \geq 0 \text{ and } x \neq 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \neq -1 \end{cases}$ , and  $\lim_{x \rightarrow -1} (1-x) = 1 - (-1) = 2$ .

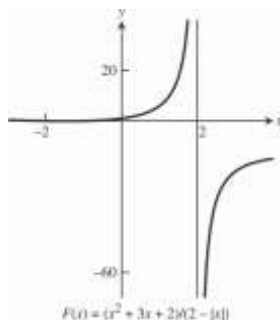
72. (a)  $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

$x$	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
$F(x)$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

$x$	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
$F(x)$	-0.9	-0.99	-0.999	-0.9999	-0.99999	-0.999999

(b)



(c)  $F(x) = \frac{x^2 + 3x + 2}{2 - |x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \geq 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \neq -2 \end{cases}$ , and  $\lim_{x \rightarrow -2} (x+1) = -2+1 = -1$ .

73. (a)  $g(\theta) = (\sin \theta)\theta$

$\theta$	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

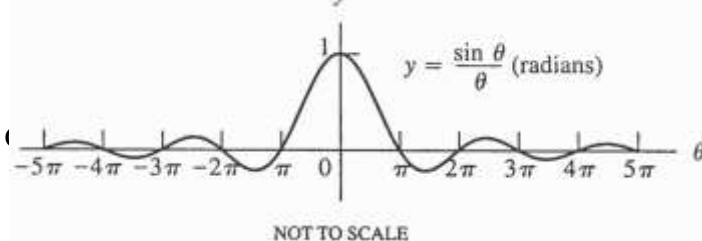
  

$\theta$	-.1	-.01	-.001	-.0001	-.00001	-.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \rightarrow 0} g(\theta) = 1$



(b)



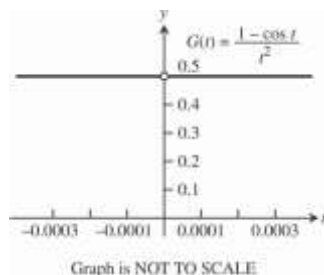
74. (a)  $G(t) = (1 - \cos t)/t^2$

$t$	.1	.01	.001	.0001	.00001	.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$t$	-.1	-.01	-.001	-.0001	-.00001	-.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$$\lim_{t \rightarrow 0} G(t) = 0.5$$

(b)



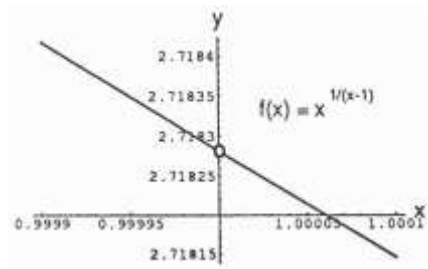
75. (a)  $f(x) = x^{1/(1-x)}$

$x$	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

$x$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$$\lim_{x \rightarrow 1} f(x) \approx 0.36788$$

(b)



Graph is NOT TO SCALE. Also, the intersection of the axes is not the origin: the axes intersect at the point (1, 2.71820).

(b)

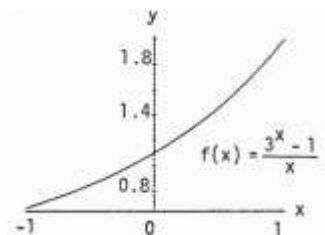
76. (a)  $f(x) = (3^x - 1)/x$

$x$	.1	.01	.001	.0001	.00001	.000001
$f(x)$	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

$x$	-.1	-.01	-.001	-.0001	-.00001	-.000001
$f(x)$	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$$\lim_{x \rightarrow 1} f(x) \approx 1.0986$$

(b)



77.  $\lim_{x \rightarrow c} f(x)$  exists at those points  $c$  where  $\lim_{x \rightarrow c} x^4 = \lim_{x \rightarrow c} x^2$ . Thus,  $c^4 = c^2 \Rightarrow c^2(1 - c^2) = 0 \Rightarrow c = 0, 1, \text{ or } -1$ .

Moreover,  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 = 0$  and  $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} f(x) = 1$ .

78. Nothing can be concluded about the values of  $f$ ,  $g$ , and  $h$  at  $x = 2$ . Yes,  $f(2)$  could be 0. Since the conditions of the sandwich theorem are satisfied,  $\lim_{x \rightarrow 2} f(x) = -5 \neq 0$ .

79.  $1 = \lim_{x \rightarrow 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x \rightarrow 4} f(x) - \lim_{x \rightarrow 4} 5}{\lim_{x \rightarrow 4} x - \lim_{x \rightarrow 4} 2} = \frac{\lim_{x \rightarrow 4} f(x) - 5}{4 - 2} \Rightarrow \lim_{x \rightarrow 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x \rightarrow 4} f(x) = 2 + 5 = 7$ .

80. (a)  $1 = \lim_{x \rightarrow -2} \frac{f(x)}{x^2} = \frac{\lim_{x \rightarrow -2} f(x)}{\lim_{x \rightarrow -2} x^2} = \frac{\lim_{x \rightarrow -2} f(x)}{4} \Rightarrow \lim_{x \rightarrow -2} f(x) = 4$ .

(b)  $1 = \lim_{x \rightarrow -2} \frac{f(x)}{x^2} = \left[ \lim_{x \rightarrow -2} \frac{f(x)}{x} \right] \left[ \lim_{x \rightarrow -2} \frac{1}{x} \right] = \left[ \lim_{x \rightarrow -2} \frac{f(x)}{x} \right] \left( -\frac{1}{2} \right) \Rightarrow \lim_{x \rightarrow -2} \frac{f(x)}{x} = -2$ .

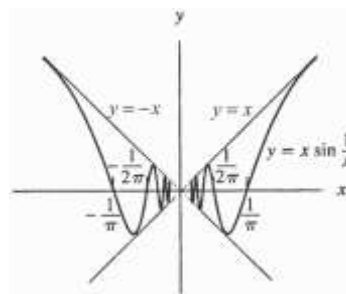
81. (a)  $0 = 3 \cdot 0 = \left[ \lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \right] \left[ \lim_{x \rightarrow 2} (x-2) \right] = \lim_{x \rightarrow 2} \left[ \left( \frac{f(x)-5}{x-2} \right) (x-2) \right] = \lim_{x \rightarrow 2} [f(x) - 5]$   
 $= \lim_{x \rightarrow 2} f(x) - 5 \Rightarrow \lim_{x \rightarrow 2} f(x) = 5$ .

(b)  $0 = 4 \cdot 0 = \left[ \lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \right] \left[ \lim_{x \rightarrow 2} (x-2) \right] \Rightarrow \lim_{x \rightarrow 2} f(x) = 5$  as in part (a).

82. (a)  $0 = 1 \cdot 0 = \left[ \lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[ \lim_{x \rightarrow 0} x^2 \right] = \left[ \lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[ \lim_{x \rightarrow 0} x^2 \right] = \lim_{x \rightarrow 0} \left[ \frac{f(x)}{x^2} \cdot x^2 \right] = \lim_{x \rightarrow 0} f(x)$ .  
 That is,  $\lim_{x \rightarrow 0} f(x) = 0$ .

(b)  $0 = 1 \cdot 0 = \left[ \lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[ \lim_{x \rightarrow 0} x \right] = \lim_{x \rightarrow 0} \left[ \frac{f(x)}{x^2} \cdot x \right] = \lim_{x \rightarrow 0} \frac{f(x)}{x}$ . That is,  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$ .

83. (a)  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

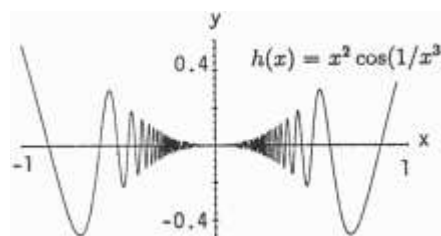


(b)  $-1 \leq \sin \frac{1}{x} \leq 1$  for  $x \neq 0$ :

$x > 0 \Rightarrow -x \leq x \sin \frac{1}{x} \leq x \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$  by the sandwich theorem;

$x < 0 \Rightarrow -x \geq x \sin \frac{1}{x} \geq x \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$  by the sandwich theorem.

84. (a)  $\lim_{x \rightarrow 0} x^2 \cos \left( \frac{1}{x^3} \right) = 0$



(b)  $-1 \leq \cos \left( \frac{1}{x^3} \right) \leq 1$  for  $x \neq 0 \Rightarrow -x^2 \leq x^2 \cos \left( \frac{1}{x^3} \right) \leq x^2 \Rightarrow \lim_{x \rightarrow 0} x^2 \cos \left( \frac{1}{x^3} \right) = 0$  by the sandwich theorem since  $\lim_{x \rightarrow 0} x^2 = 0$ .

85–90. Example CAS commands:

Maple:

```
f := x -> (x^4 - 16)/(x - 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
      title = "Section 2.2, #85(a)" );
limit( f(x), x = x0 );
```

In Exercise 87, note that the standard cube root,  $x^{1/3}$ , is not defined for  $x < 0$  in many CASs. This can be overcome in Maple by entering the function as  $f := x \rightarrow (\text{surd}(x+1, 3) - 1)/x$ .

Mathematica: (assigned function and values for  $x_0$  and  $h$  may vary)

```
Clear[f, x]
f[x_]:= (x^3 - x^2 - 5x - 3)/(x + 1)^2
x0 = -1; h = 0.1;
Plot[f[x], {x, x0 - h, x0 + h}]
Limit[f[x], x -> x0]
```

## 2.3 THE PRECISE DEFINITION OF A LIMIT



Step 1:  $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2:  $\delta + 5 = 7 \Rightarrow \delta = 2$ , or  $-\delta + 5 = 1 \Rightarrow \delta = 4$ .

The value of  
 $\delta$  which  
 assures  $|x - 5| < \delta \Rightarrow 1 < x < 7$  is the  
 smaller  
 value,  $\delta = 2$ .

| |



Step 1:  $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$

Step 2:  $-\delta+2=1 \Rightarrow \delta=1$ , or  $\delta+2=7 \Rightarrow \delta=5$ .

The value of  $\delta$  which assures  $|x-2| < \delta \Rightarrow 1 < x < 7$  is the smaller value,  $\delta=1$ .



Step 1:  $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$

Step 2:  $-\delta-3=-\frac{7}{2} \Rightarrow \delta=\frac{1}{2}$ , or  $\delta-3=-\frac{1}{2} \Rightarrow \delta=\frac{5}{2}$ .

The value of  $\delta$  which assures  $|x-(-3)| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$  is the smaller value,  $\delta=\frac{1}{2}$ .



Step 1:  $|x-(-\frac{3}{2})| < \delta \Rightarrow -\delta < x+\frac{3}{2} < \delta \Rightarrow -\delta-\frac{3}{2} < x < \delta-\frac{3}{2}$

Step 2:  $-\delta-\frac{3}{2}=-\frac{7}{2} \Rightarrow \delta=2$ , or  $\delta-\frac{3}{2}=-\frac{1}{2} \Rightarrow \delta=1$ .

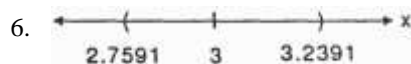
The value of  $\delta$  which assures  $|x-(-\frac{3}{2})| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$  is the smaller value,  $\delta=1$ .



Step 1:  $|x-\frac{1}{2}| < \delta \Rightarrow -\delta < x-\frac{1}{2} < \delta \Rightarrow -\delta+\frac{1}{2} < x < \delta+\frac{1}{2}$

Step 2:  $-\delta+\frac{1}{2}=\frac{4}{9} \Rightarrow \delta=\frac{1}{18}$ , or  $\delta+\frac{1}{2}=\frac{4}{7} \Rightarrow \delta=\frac{1}{14}$ .

The value of  $\delta$  which assures  $|x-\frac{1}{2}| < \delta \Rightarrow \frac{4}{9} < x < \frac{4}{7}$  is the smaller value,  $\delta=\frac{1}{18}$ .



Step 1:  $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$

Step 2:  $-\delta+3=2.7591 \Rightarrow \delta=0.2409$ , or  $\delta+3=3.2391 \Rightarrow \delta=0.2391$ .

The value of  $\delta$  which assures  $|x-3| < \delta \Rightarrow 2.7591 < x < 3.2391$  is the smaller value,  $\delta=0.2391$ .

7. Step 1:  $|x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$

Step 2: From the graph,  $-\delta+5=4.9 \Rightarrow \delta=0.1$ , or  $\delta+5=5.1 \Rightarrow \delta=0.1$ ; thus  $\delta=0.1$  in either case.

8. Step 1:  $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$

Step 2: From the graph,  $-\delta-3=-3.1 \Rightarrow \delta=0.1$ , or  $\delta-3=-2.9 \Rightarrow \delta=0.1$ ; thus  $\delta=0.1$ .

9. Step 1:  $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1$

Step 2: From the graph,  $-\delta+1=\frac{9}{16} \Rightarrow \delta=\frac{7}{16}$ , or  $\delta+1=\frac{25}{16} \Rightarrow \delta=\frac{7}{16}$ ; thus  $\delta=\frac{7}{16}$ .

10. Step 1:  $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$

Step 2: From the graph,  $-\delta+3=2.61 \Rightarrow \delta=0.39$ , or  $\delta+3=3.41 \Rightarrow \delta=0.41$ ; thus  $\delta=0.39$ .

11. Step 1:  $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$   
 Step 2: From the graph,  $-\delta+2 = \sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$ , or  $\delta+2 = \sqrt{5} \Rightarrow \delta = \sqrt{5}-2 \approx 0.2361$ ; thus  $\delta = \sqrt{5}-2$ .
12. Step 1:  $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$   
 Step 2: From the graph,  $-\delta-1 = -\frac{\sqrt{5}}{2} \Rightarrow \delta = \frac{\sqrt{5}-2}{2} \approx 0.118$  or  $\delta-1 = -\frac{\sqrt{3}}{2} \Rightarrow \delta = \frac{2-\sqrt{3}}{2} \approx 0.1340$ ; thus  $\delta = \frac{\sqrt{5}-2}{2}$ .
13. Step 1:  $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$   
 Step 2: From the graph,  $-\delta-1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$ , or  $\delta-1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$ ; thus  $\delta = \frac{9}{25} = 0.36$ .
14. Step 1:  $|x-\frac{1}{2}| < \delta \Rightarrow -\delta < x-\frac{1}{2} < \delta \Rightarrow -\delta+\frac{1}{2} < x < \delta+\frac{1}{2}$   
 Step 2: From the graph,  $-\delta+\frac{1}{2} = \frac{1}{2.01} \Rightarrow \delta = \frac{1}{2} - \frac{1}{2.01} \approx 0.00248$ , or  $\delta+\frac{1}{2} = \frac{1}{1.99} \Rightarrow \delta = \frac{1}{1.99} - \frac{1}{2} \approx 0.00251$ ; thus  $\delta = 0.00248$ .
15. Step 1:  $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$   
 Step 2:  $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta = 0.01$ .
16. Step 1:  $|(2x-2)-(-6)| < 0.02 \Rightarrow |2x+4| < 0.02 \Rightarrow -0.02 < 2x+4 < 0.02 \Rightarrow -4.02 < 2x < -3.98 \Rightarrow -2.01 < x < -1.99$   
 Step 2:  $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2 \Rightarrow \delta = 0.01$ .
17. Step 1:  $x+1-1 < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$   
 $\Rightarrow -0.19 < x < 0.21$   
 Step 2:  $|x-0| < \delta \Rightarrow -\delta < x < \delta$ . Then,  $-\delta = -0.19 \Rightarrow \delta = 0.19$  or  $\delta = 0.21$ ; thus,  $\delta = 0.19$ .
18. Step 1:  $|\sqrt{x}-\frac{1}{2}| < 0.1 \Rightarrow -0.1 < \sqrt{x}-\frac{1}{2} < 0.1 \Rightarrow 0.4 < \sqrt{x} < 0.6 \Rightarrow 0.16 < x < 0.36$   
 Step 2:  $|x-\frac{1}{4}| < \delta \Rightarrow -\delta < x-\frac{1}{4} < \delta \Rightarrow$   
 Then  $-\delta+\frac{1}{4} = 0.16 \Rightarrow \delta = 0.09$  or  $\delta+\frac{1}{4} = 0.36 \Rightarrow \delta = 0.11$ ; thus  $\delta = 0.09$ .
19. Step 1:  $|\sqrt{19-x}-3| < 1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$   
 $\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$  or  $3 < x < 15$   
 Step 2:  $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10$ .  
 Then  $-\delta+10 = 3 \Rightarrow \delta = 7$ , or  $\delta+10 = 15 \Rightarrow \delta = 5$ ; thus  $\delta = 5$ .
20. Step 1:  $|\sqrt{x-7}-4| < 1 \Rightarrow -1 < \sqrt{x-7}-4 < 1 \Rightarrow 3 < \sqrt{x-7} < 5 \Rightarrow 9 < x-7 < 25 \Rightarrow 16 < x < 32$   
 Step 2:  $|x-23| < \delta \Rightarrow -\delta < x-23 < \delta \Rightarrow -\delta+23 < x < \delta+23$ .  
 Then  $-\delta+23 = 16 \Rightarrow \delta = 7$ , or  $\delta+23 = 32 \Rightarrow \delta = 9$ ; thus  $\delta = 7$ .
21. Step 1:  $|\frac{1}{x}-\frac{1}{4}| < 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$  or  $\frac{10}{3} < x < 5$ .  
 Step 2:  $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$ .  
 Then  $-\delta+4 = \frac{10}{3}$  or  $\delta = \frac{2}{3}$ , or  $\delta+4 = 5$  or  $\delta = 1$ ; thus  $\delta = \frac{2}{3}$ .

22. Step 1:  $|x^2 - 3| < 0.1 \Rightarrow -0.1 < x^2 - 3 < 0.1 \Rightarrow 2.9 < x^2 < 3.1 \Rightarrow \sqrt{2.9} < x < \sqrt{3.1}$   
 Step 2:  $|x - \sqrt{3}| < \delta \Rightarrow -\delta < x - \sqrt{3} < \delta \Rightarrow -\delta + \sqrt{3} < x < \delta + \sqrt{3}$ .  
 Then  $-\delta + \sqrt{3} = \sqrt{2.9} \Rightarrow \delta = \sqrt{3} - \sqrt{2.9} \approx 0.0291$ , or  $\delta + \sqrt{3} = \sqrt{3.1} \Rightarrow \delta = \sqrt{3.1} - \sqrt{3} \approx 0.0286$ ;  
 thus  $\delta = 0.0286$ .
23. Step 1:  $|x^2 - 4| < 0.5 \Rightarrow -0.5 < x^2 - 4 < 0.5 \Rightarrow 3.5 < x^2 < 4.5 \Rightarrow \sqrt{3.5} < x < \sqrt{4.5} \Rightarrow -\sqrt{4.5} < x < -\sqrt{3.5}$ ,  
 for  $x$  near  $-2$ .  
 Step 2:  $|x - (-2)| < \delta \Rightarrow -\delta < x + 2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2$ .  
 Then  $-\delta - 2 = -\sqrt{4.5} \Rightarrow \delta = \sqrt{4.5} - 2 \approx 0.1213$ , or  $\delta - 2 = -\sqrt{3.5} \Rightarrow \delta = 2 - \sqrt{3.5} \approx 0.1292$ ;  
 thus  $\delta = \sqrt{4.5} - 2 \approx 0.12$ .
24. Step 1:  $|\frac{1}{x} - (-1)| < 0.1 \Rightarrow -0.1 < \frac{1}{x} + 1 < 0.1 \Rightarrow -\frac{11}{10} < \frac{1}{x} < -\frac{9}{10} \Rightarrow -\frac{10}{11} > x > -\frac{10}{9}$  or  $-\frac{10}{9} < x < -\frac{10}{11}$   
 Step 2:  $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$ .  
 Then  $-\delta - 1 = -\frac{10}{9} \Rightarrow \delta = \frac{1}{9}$ , or  $\delta - 1 = -\frac{10}{11} \Rightarrow \delta = \frac{1}{11}$ ; thus  $\delta = \frac{1}{11}$ .
25. Step 1:  $|(x^2 - 5) - 11| < 1 \Rightarrow |x^2 - 16| < 1 \Rightarrow -1 < x^2 - 16 < 1 \Rightarrow 15 < x^2 < 17 \Rightarrow \sqrt{15} < x < \sqrt{17}$ .  
 Step 2:  $|x - 4| < \delta \Rightarrow -\delta < x - 4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4$ .  
 Then  $-\delta + 4 = \sqrt{15} \Rightarrow \delta = 4 - \sqrt{15} \approx 0.1270$ , or  $\delta + 4 = \sqrt{17} \Rightarrow \delta = \sqrt{17} - 4 \approx 0.1231$ ; thus  
 $\delta = \sqrt{17} - 4 \approx 0.12$ .
26. Step 1:  $|\frac{120}{x} - 5| < 1 \Rightarrow -1 < \frac{120}{x} - 5 < 1 \Rightarrow 4 < \frac{120}{x} < 6 \Rightarrow \frac{1}{4} > \frac{x}{120} > \frac{1}{6} \Rightarrow 30 > x > 20$  or  $20 < x < 30$ .  
 Step 2:  $|x - 24| < \delta \Rightarrow -\delta < x - 24 < \delta \Rightarrow -\delta + 24 < x < \delta + 24$ .  
 Then  $-\delta + 24 = 20 \Rightarrow \delta = 4$ , or  $\delta + 24 = 30 \Rightarrow \delta = 6$ ; thus  $\delta = 4$ .
27. Step 1:  $|mx - 2m| < 0.03 \Rightarrow -0.03 < mx - 2m < 0.03 \Rightarrow -0.03 + 2m < mx < 0.03 + 2m \Rightarrow 2 - \frac{0.03}{m} < x < 2 + \frac{0.03}{m}$ .  
 Step 2:  $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$ .  
 Then  $-\delta + 2 = 2 - \frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$ , or  $\delta + 2 = 2 + \frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$ . In either case,  $\delta = \frac{0.03}{m}$ .
28. Step 1:  $|mx - 3m| < c \Rightarrow -c < mx - 3m < c \Rightarrow -c + 3m < mx < c + 3m \Rightarrow 3 - \frac{c}{m} < x < 3 + \frac{c}{m}$ .  
 Step 2:  $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$ .  
 Then  $-\delta + 3 = 3 - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$ , or  $\delta + 3 = 3 + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$ . In either case,  $\delta = \frac{c}{m}$ .
29. Step 1:  $|(mx + b) - (\frac{2}{m} + b)| < c \Rightarrow -c < mx - \frac{2}{m} < c \Rightarrow -c + \frac{2}{m} < mx < c + \frac{2}{m} \Rightarrow \frac{2}{m} - \frac{c}{m} < x < \frac{2}{m} + \frac{c}{m}$ .  
 Step 2:  $|x - \frac{1}{2}| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$ .  
 Then  $-\delta + \frac{1}{2} = \frac{2}{m} - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$ , or  $\delta + \frac{1}{2} = \frac{2}{m} + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$ . In either case,  $\delta = \frac{c}{m}$ .
30. Step 1:  $|(mx + b) - (m + b)| < 0.05 \Rightarrow -0.05 < mx - m < 0.05 \Rightarrow -0.05 + m < mx < 0.05 + m$   
 $\Rightarrow 1 - \frac{0.05}{m} < x < 1 + \frac{0.05}{m}$ .  
 Step 2:  $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$ .  
 Then  $-\delta + 1 = 1 - \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$ , or  $\delta + 1 = 1 + \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$ . In either case,  $\delta = \frac{0.05}{m}$ .
31.  $\lim_{x \rightarrow 3} (3 - 2x) = 3 - 2(3) = -3$   
 Step 1:  $|(3 - 2x) - (-3)| < 0.02 \Rightarrow -0.02 < 6 - 2x < 0.02 \Rightarrow -6.02 < -2x < -5.98 \Rightarrow 3.01 > x > 2.99$  or  
 $2.99 < x < 3.01$ .

Step 2:  $0 < |x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$ .

Then  $-\delta+3 = 2.99 \Rightarrow \delta = 0.01$ , or  $\delta+3 = 3.01 \Rightarrow \delta = 0.01$ ; thus  $\delta = 0.01$ .

32.  $\lim_{x \rightarrow -1} (-3x-2) = (-3)(-1)-2 = 1$

Step 1:  $|(-3x-2)-1| < 0.03 \Rightarrow -0.03 < -3x-3 < 0.03 \Rightarrow 0.01 > x+1 > -0.01 \Rightarrow -1.01 < x < -0.99$ .

Step 2:  $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$ .

Then  $-\delta-1 = -1.01 \Rightarrow \delta = 0.01$ , or  $\delta-1 = -0.99 \Rightarrow \delta = 0.01$ ; thus  $\delta = 0.01$ .

33.  $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \rightarrow 2} (x+2) = 2+2 = 4, x \neq 2$

Step 1:  $\left| \left( \frac{x^2-4}{x-2} \right) - 4 \right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)} - 4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05, x \neq 2$   
 $\Rightarrow 1.95 < x < 2.05, x \neq 2$ .

Step 2:  $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$ .

Then  $-\delta+2 = 1.95 \Rightarrow \delta = 0.05$ , or  $\delta+2 = 2.05 \Rightarrow \delta = 0.05$ ; thus  $\delta = 0.05$ .

34.  $\lim_{x \rightarrow -5} \frac{x^2+6x+5}{x+5} = \lim_{x \rightarrow -5} \frac{(x+5)(x+1)}{(x+5)} = \lim_{x \rightarrow -5} (x+1) = -4, x \neq -5$ .

Step 1:  $\left| \left( \frac{x^2+6x+5}{x+5} \right) - (-4) \right| < 0.05 \Rightarrow -0.05 < \frac{(x+5)(x+1)}{(x+5)} + 4 < 0.05 \Rightarrow -4.05 < x+1 < -3.95, x \neq -5$   
 $\Rightarrow -5.05 < x < -4.95, x \neq -5$ .

Step 2:  $|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta-5 < x < \delta-5$ .

Then  $-\delta-5 = -5.05 \Rightarrow \delta = 0.05$ , or  $\delta-5 = -4.95 \Rightarrow \delta = 0.05$ ; thus  $\delta = 0.05$ .

35.  $\lim_{x \rightarrow -3} \sqrt{1-5x} = \sqrt{1-5(-3)} = \sqrt{16} = 4$

Step 1:  $|\sqrt{1-5x}-4| < 0.5 \Rightarrow -0.5 < \sqrt{1-5x}-4 < 0.5 \Rightarrow 3.5 < \sqrt{1-5x} < 4.5 \Rightarrow 12.25 < 1-5x < 20.25$   
 $\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < 2.25$ .

Step 2:  $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$ .

Then  $-\delta-3 = -3.85 \Rightarrow \delta = 0.85$ , or  $\delta-3 = -2.25 \Rightarrow 0.75$ ; thus  $\delta = 0.75$ .

36.  $\lim_{x \rightarrow 2} \frac{4}{x} = \frac{4}{2} = 2$

Step 1:  $\left| \frac{4}{x} - 2 \right| < 0.4 \Rightarrow -0.4 < \frac{4}{x} - 2 < 0.4 \Rightarrow 1.6 < \frac{4}{x} < 2.4 \Rightarrow \frac{10}{16} > \frac{x}{4} > \frac{10}{24} \Rightarrow \frac{10}{4} > x > \frac{10}{6}$  or  $\frac{5}{3} < x < \frac{5}{2}$

Step 2:  $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$ .

Then  $-\delta+2 = \frac{5}{3} \Rightarrow \delta = \frac{1}{3}$ , or  $\delta+2 = \frac{5}{2} \Rightarrow \delta = \frac{1}{2}$ ; thus  $\delta = \frac{1}{3}$ .

37. Step 1:  $|(9-x)-5| < \square \Rightarrow -\square < 4-x < \square \Rightarrow -\square-4 < -x < \square-4 \Rightarrow \square+4 > x > 4-\square \Rightarrow 4-\square < x < 4+\square$

Step 2:  $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$ .

Then  $-\delta+4 = -\square+4 \Rightarrow \delta = \square$ , or  $\delta+4 = \square+4 \Rightarrow \delta = \square$ . Thus choose  $\delta = \square$ .

38. Step 1:  $|(3x-7)-2| < \square \Rightarrow -\square < 3x-9 < \square \Rightarrow 9-\square < 3x < 9+\square \Rightarrow 3-\frac{\square}{3} < x < 3+\frac{\square}{3}$ .

Step 2:  $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$ .

Then  $-\delta+3 = 3-\frac{\square}{3} \Rightarrow \delta = \frac{\square}{3}$ , or  $\delta+3 = 3+\frac{\square}{3} \Rightarrow \delta = \frac{\square}{3}$ . Thus choose  $\delta = \frac{\square}{3}$ .

39. Step 1:  $|\sqrt{x-5}-2| < \square \Rightarrow -\square < \sqrt{x-5}-2 < \square \Rightarrow 2-\square < \sqrt{x-5} < 2+\square \Rightarrow (2-\square)^2 < x-5 < (2+\square)^2$   
 $\Rightarrow (2-\square)^2 + 5 < x < (2+\square)^2 + 5$ .

Step 2:  $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta+9 < x < \delta+9$ .



Then  $-\delta + 9$

$$= \frac{\epsilon^2}{4} - 4\epsilon +$$

$$9 \Rightarrow \delta = 4\epsilon$$

$$- \frac{\epsilon^2}{4}, \text{ or } \delta +$$

$$9 = \frac{\epsilon^2}{4} + 4\epsilon$$

$$+ 9 \Rightarrow \delta =$$

$$4\epsilon + \frac{\epsilon^2}{4}.$$

Thus

choose the

smaller

distance,  $\delta$

$$= 4\epsilon - \frac{\epsilon^2}{4}.$$

40. Step 1:  $|\sqrt{4-x}-2| < \square \Rightarrow -\square < \sqrt{4-x}-2 < \square \Rightarrow 2-\square < \sqrt{4-x} < 2+\square \Rightarrow (2-\square)^2 < 4-x < (2+\square)^2$   
 $\Rightarrow -(2+\square)^2 < x-4 < -(2-\square)^2 \Rightarrow -(2+\square)^2+4 < x < -(2-\square)^2+4.$   
 Step 2:  $|x-0| < \delta \Rightarrow -\delta < x < \delta.$   
 Then  $-\delta = -(2+\square)^2+4 = -\square^2-4\square \Rightarrow \delta = 4\square+\square^2$ , or  $\delta = -(2-\square)^2+4 = 4\square-\square^2$ . Thus choose the smaller distance,  $\delta = 4\square-\square^2$ .
41. Step 1: For  $x \neq 1$ ,  $|x^2-1| < \square \Rightarrow -\square < x^2-1 < \square \Rightarrow 1-\square < x^2 < 1+\square \Rightarrow \sqrt{1-\square} < |x| < \sqrt{1+\square}$   
 $\Rightarrow \sqrt{1-\square} < x < \sqrt{1+\square}$  near  $x=1$ .  
 Step 2:  $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1.$   
 Then  $-\delta+1 = \sqrt{1-\square} \Rightarrow \delta = 1-\sqrt{1-\square}$  or  $\delta+1 = \sqrt{1+\square} \Rightarrow \delta = \sqrt{1+\square}-1$ . Choose  $\delta = \min\{1-\sqrt{1-\square}, \sqrt{1+\square}-1\}$ , that is, the smaller of the two distances.
42. Step 1: For  $x \neq -2$ ,  $|x^2-4| < \square \Rightarrow -\square < x^2-4 < \square \Rightarrow 4-\square < x^2 < 4+\square \Rightarrow \sqrt{4-\square} < |x| < \sqrt{4+\square} \Rightarrow -\sqrt{4+\square} < x < -\sqrt{4-\square}$  near  $x=-2$ .  
 Step 2:  $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2.$   
 Then  $-\delta-2 = -\sqrt{4+\square} \Rightarrow \delta = \sqrt{4+\square}-2$ , or  $\delta-2 = -\sqrt{4-\square} \Rightarrow \delta = 2-\sqrt{4-\square}$ . Choose  $\delta = \min\{\sqrt{4+\square}-2, 2-\sqrt{4-\square}\}$ .
43. Step 1:  $|\frac{1}{x}-1| < \square \Rightarrow -\square < \frac{1}{x}-1 < \square \Rightarrow 1-\square < \frac{1}{x} < 1+\square \Rightarrow \frac{1}{1+\square} < x < \frac{1}{1-\square}$   
 Step 2:  $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$   
 Then  $1-\delta = \frac{1}{1+\square} \Rightarrow \delta = 1-\frac{1}{1+\square} = \frac{\square}{1+\square}$ , or  $1+\delta = \frac{1}{1-\square} \Rightarrow \delta = \frac{1}{1-\square}-1 = \frac{\square}{1-\square}$ . Choose  $\delta = \frac{\square}{1+\square}$ , the smaller of the two distances.
44. Step 1:  $|\frac{1}{x^2}-\frac{1}{3}| < \square \Rightarrow -\square < \frac{1}{x^2}-\frac{1}{3} < \square \Rightarrow \frac{1}{3}-\square < \frac{1}{x^2} < \frac{1}{3}+\square \Rightarrow \frac{1-3\square}{3} < \frac{1}{x^2} < \frac{1+3\square}{3}$   
 $\Rightarrow \frac{3}{1+3\square} > x^2 > \frac{3}{1-3\square} \Rightarrow \sqrt{\frac{3}{1+3\square}} < x < \sqrt{\frac{3}{1-3\square}}$  or  $\sqrt{\frac{3}{1+3\square}} < x < \sqrt{\frac{3}{1-3\square}}$  for  $x$  near  $\sqrt{3}$ .  
 Step 2:  $|x-\sqrt{3}| < \delta \Rightarrow -\delta < x-\sqrt{3} < \delta \Rightarrow \sqrt{3}-\delta < x < \sqrt{3}+\delta.$   
 Then  $\sqrt{3}-\delta = \sqrt{\frac{3}{1+3\square}} \Rightarrow \delta = \sqrt{3}-\sqrt{\frac{3}{1+3\square}}$ , or  $\sqrt{3}+\delta = \sqrt{\frac{3}{1-3\square}} \Rightarrow \delta = \sqrt{\frac{3}{1-3\square}}-\sqrt{3}$ . Choose  $\delta = \min\{\sqrt{3}-\sqrt{\frac{3}{1+3\square}}, \sqrt{\frac{3}{1-3\square}}-\sqrt{3}\}$ .
45. Step 1:  $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right| < \square \Rightarrow -\square < (x-3)+6 < \square, x \neq -3 \Rightarrow -\square < x+3 < \square \Rightarrow -\square-3 < x < \square-3.$   
 Step 2:  $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3.$   
 Then  $-\delta-3 = -\square-3 \Rightarrow \delta = \square$  or  $\delta-3 = \square-3 \Rightarrow \delta = \square$ . Choose  $\delta = \square$ .
46. Step 1:  $\left|\left(\frac{x^2-1}{x-1}\right)-2\right| < \square \Rightarrow -\square < (x+1)-2 < \square, x \neq 1 \Rightarrow 1-\square < x < 1+\square$   
 Step 2:  $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$   
 Then  $1-\delta = 1-\square \Rightarrow \delta = \square$ , or  $1+\delta = 1+\square \Rightarrow \delta = \square$ . Choose  $\delta = \square$ .
47. Step 1:  $x < 1: |(4-2x)-2| < \square \Rightarrow 0 < 2-2x < \square$  since  $x < 1$ . Thus,  $1-\frac{\square}{2} < x < 0$ ;  
 $x \geq 1: |(6x-4)-2| < \square \Rightarrow 0 \leq 6x-6 < \square$  since  $x \geq 1$ . Thus,  $1 \leq x < 1+\frac{\square}{6}$ .  
 Step 2:  $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$   
 Then  $1-\delta = 1-\frac{\square}{2} \Rightarrow \delta = \frac{\square}{2}$ , or  $1+\delta = 1+\frac{\square}{6} \Rightarrow \delta = \frac{\square}{6}$ . Choose  $\delta = \frac{\square}{6}$ .

48. Step 1:  $x < 0: |2x - 0| < \epsilon \Rightarrow -\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0;$

$x \geq 0: |x - 0| < \epsilon \Rightarrow 0 \leq x < \epsilon$

Step 2:  $|x - 0| < \delta \Rightarrow -\delta < x < \delta.$

Then  $-\delta = -\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$ , or  $\delta = 2\epsilon \Rightarrow \delta = 2\epsilon$ . Choose  $\delta = \frac{\epsilon}{2}$ .

49. By the figure,  $-x \leq x \sin \frac{1}{x} \leq x$  for all  $x > 0$  and  $-x \geq x \sin \frac{1}{x} \geq x$  for  $x < 0$ . Since  $\lim_{x \rightarrow 0} (-x) = \lim_{x \rightarrow 0} x = 0$ , then by the sandwich theorem, in either case,  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ .

50. By the figure,  $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$  for all  $x$  except possibly at  $x = 0$ . Since  $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$ , then by the sandwich theorem,  $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$ .

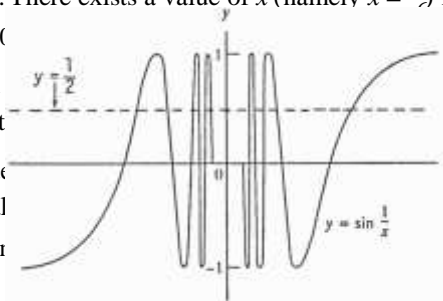
51. As  $x$  approaches the value 0, the values of  $g(x)$  approach  $k$ . Thus for every number  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $0 < |x - 0| < \delta \Rightarrow |g(x) - k| < \epsilon$

52. Write  $x = h + c$ . Then  $0 < |x - c| < \delta \Leftrightarrow -\delta < x - c < \delta, x \neq c \Leftrightarrow -\delta < (h + c) - c < \delta, h + c \neq c \Leftrightarrow -\delta < h < \delta, h \neq 0 \Leftrightarrow 0 < |h - 0| < \delta$ .

Thus,  $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow$  for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  whenever  $0 < |x - c| < \delta \Leftrightarrow |f(h + c) - L| < \epsilon$  whenever  $0 < |h - 0| < \delta \Leftrightarrow \lim_{h \rightarrow 0} f(h + c) = L$ .

53. Let  $f(x) = x^2$ . The function values do get closer to  $-1$  as  $x$  approaches 0, but  $\lim_{x \rightarrow 0} f(x) = 0$ , not  $-1$ . The function  $f(x) = x^2$  never gets arbitrarily close to  $-1$  for  $x$  near 0.

54. Let  $f(x) = \sin x, L = \frac{1}{2}$ , and  $x_0 = 0$ . There exists a value of  $x$  (namely  $x = \frac{\pi}{6}$ ) for which  $|\sin x - \frac{1}{2}| < \epsilon$  for any given  $\epsilon > 0$ . However,  $\lim_{x \rightarrow 0} \sin x = 0$ .  
 As another example, let  $g(x) = \sin \frac{1}{x}$ . No matter how close we get to  $x = 0$ , there are infinitely many values of  $x$  near 0 such that  $\sin \frac{1}{x} = \frac{1}{2}$ . The wrong statement does not require all values of  $x$  to satisfy the inequality. As you can see from the figure that there are always values of  $x$  near 0 such that  $\sin \frac{1}{x} = \frac{1}{2}$ . Again you can see from the figure that there are always values of  $x$  near 0 such that  $\sin \frac{1}{x} = 0$ . If we choose  $\epsilon < \frac{1}{4}$ , we cannot satisfy the inequality  $|\sin \frac{1}{x} - \frac{1}{2}| < \epsilon$  for  $x_0 = 0$ .



55.  $|A - 9| \leq 0.01 \Rightarrow -0.01 \leq \pi \left( \frac{x}{2} \right)^2 - 9 \leq 0.01 \Rightarrow 8.99 \leq \frac{x^2}{4} \leq 9.01 \Rightarrow (8.99) \leq x^2 \leq (9.01)$

$\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \leq x \leq 2\sqrt{\frac{9.01}{\pi}}$  or  $3.384 \leq x \leq 3.387$ . To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

$$56. V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left| \frac{V}{R} - 5 \right| \leq 0.1 \Rightarrow -0.1 \leq \frac{120}{R} - 5 \leq 0.1 \Rightarrow 4.9 \leq \frac{120}{R} \leq 5.1 \Rightarrow \frac{10}{49} \geq \frac{R}{120} \geq \frac{10}{51} \\ \Rightarrow \frac{(120)(10)}{51} \leq R \leq \frac{(120)(10)}{49} \Rightarrow 23.53 \leq R \leq 24.48.$$

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a)  $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$ . Then  $|f(x)-2| = |x-2| = 2-x > 2-1 = 1$ . That is,  $|f(x)-2| \geq 1 \geq \frac{1}{2}$  no matter how small  $\delta$  is taken when  $1-\delta < x < 1 \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 2$ .
- (b)  $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$ . Then  $|f(x)-1| = |(x+1)-1| = |x| = x > 1$ . That is,  $|f(x)-1| \geq 1$  no matter how small  $\delta$  is taken when  $1 < x < 1+\delta \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 1$ .
- (c)  $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$ . Then  $|f(x)-1.5| = |x-1.5| = 1.5-x > 1.5-1 = 0.5$ . Also,  $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$ . Then  $|f(x)-1.5| = |(x+1)-1.5| = |x-0.5| = x-0.5 > 1-0.5 = 0.5$ . Thus, no matter how small  $\delta$  is taken, there exists a value of  $x$  such that  $-\delta < x-1 < \delta$  but  $|f(x)-1.5| \geq \frac{1}{2} \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 1.5$ .
58. (a) For  $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-4| = 2$ . Thus for  $\square < 2$ ,  $|h(x)-4| \geq \square$  whenever  $2 < x < 2+\delta$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 4$ .
- (b) For  $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-3| = 1$ . Thus for  $\square < 1$ ,  $|h(x)-3| \geq \square$  whenever  $2 < x < 2+\delta$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 3$ .
- (c) For  $2-\delta < x < 2 \Rightarrow h(x) = x^2$  so  $|h(x)-2| = |x^2-2|$ . No matter how small  $\delta > 0$  is chosen,  $x^2$  is close to 4 when  $x$  is near 2 and to the left on the real line  $\Rightarrow |x^2-2|$  will be close to 2. Thus if  $\square < 1$ ,  $|h(x)-2| \geq \square$  whenever  $2-\delta < x < 2$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 2$ .
59. (a) For  $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-4| \geq 0.8$ . Thus for  $\square < 0.8$ ,  $|f(x)-4| \geq \square$  whenever  $3-\delta < x < 3$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 4$ .
- (b) For  $3 < x < 3+\delta \Rightarrow f(x) < 3 \Rightarrow |f(x)-4.8| \geq 1.8$ . Thus for  $\square < 1.8$ ,  $|f(x)-4.8| \geq \square$  whenever  $3 < x < 3+\delta$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 4.8$ .
- (c) For  $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-3| \geq 1.8$ . Again, for  $\square < 1.8$ ,  $|f(x)-3| \geq \square$  whenever  $3-\delta < x < 3$  no matter how small we choose  $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 3$ .
60. (a) No matter how small we choose  $\delta > 0$ , for  $x$  near  $-1$  satisfying  $-1-\delta < x < -1+\delta$ , the values of  $g(x)$  are near  $1 \Rightarrow |g(x)-2|$  is near  $1$ . Then, for  $\square = \frac{1}{2}$  we have  $|g(x)-2| \geq \frac{1}{2}$  for some  $x$  satisfying  $-1-\delta < x < -1+\delta$ , or  $0 < |x+1| < \delta \Rightarrow \lim_{x \rightarrow -1} g(x) \neq 2$ .
- (b) Yes,  $\lim_{x \rightarrow -1} g(x) = 1$  because from the graph we can find a  $\delta > 0$  such that  $|g(x)-1| < \square$  if  $0 < |x-(-1)| < \delta$ .

61–66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3); x0 := 3;
```

```
plot( f(x), x=x0-1..x0+1, color=black, # (a)
```

```
title="Section 2.3, #61(a)");
```

```
L := limit( f(x), x=x0 ); # (b)
```

```
epsilon := 0.2; # (c)
```

```
plot( [f(x), L-epsilon, L+epsilon], x=x0-0.01..x0+0.01,
color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)");
```

```

q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1 );      # (d)
delta := abs(x0-q);
plot( [f(x), L-epsilon, L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)");
for eps in [0.1, 0.005, 0.001 ] do                          # (e)
q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1 );
delta := abs(x0-q);
head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f\n", eps, delta );
print(plot( [f(x), L-eps, L+eps], x=x0-delta..x0+delta,
color=black, linestyle=[1,3,3], title=head ));
end do:

```

Mathematica (assigned function and values for x0, eps and del may vary):

```

Clear[f, x]
y1:= L - eps; y2:= L + eps; x0 = 1;
f[x_]:= (3x^2 - (7x + 1)Sqrt[x] + 5)/(x - 1)
Plot[f[x], {x, x0 - 0.2, x0 + 0.2}]
L:= Limit[f[x], x -> x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2}, {x, x0 - del, x0 + del}, PlotRange -> {L - 2eps, L + 2eps}]

```

## 2.4 ONE-SIDED LIMITS

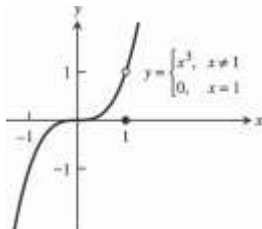
- |           |           |           |           |
|-----------|-----------|-----------|-----------|
| (a) True  | (b) True  | (c) False | (d) True  |
| (e) True  | (f) True  | (g) False | (h) False |
| (i) False | (j) False | (k) True  | (l) False |
- |          |           |           |          |
|----------|-----------|-----------|----------|
| (a) True | (b) False | (c) False | (d) True |
| (e) True | (f) True  | (g) True  | (h) True |
| (i) True | (j) False | (k) True  |          |
- |                                                                                                                                 |
|---------------------------------------------------------------------------------------------------------------------------------|
| (a) $\lim_{x \rightarrow 2^+} f(x) = \frac{2}{2} + 1 = 2$ , $\lim_{x \rightarrow 2^-} f(x) = 3 - 2 = 1$                         |
| (b) No, $\lim_{x \rightarrow 2} f(x)$ does not exist because $\lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$ |
| (c) $\lim_{x \rightarrow 4^-} f(x) = \frac{4}{2} + 1 = 3$ , $\lim_{x \rightarrow 4^+} f(x) = \frac{4}{2} + 1 = 3$               |
| (d) Yes, $\lim_{x \rightarrow 4} f(x) = 3$ because $3 = \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x)$          |
- |                                                                                                                           |
|---------------------------------------------------------------------------------------------------------------------------|
| (a) $\lim_{x \rightarrow 2^+} f(x) = \frac{2}{2} = 1$ , $\lim_{x \rightarrow 2^-} f(x) = 3 - 2 = 1$ , $f(2) = 2$          |
| (b) Yes, $\lim_{x \rightarrow 2} f(x) = 1$ because $1 = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$    |
| (c) $\lim_{x \rightarrow -1^-} f(x) = 3 - (-1) = 4$ , $\lim_{x \rightarrow -1^+} f(x) = 3 - (-1) = 4$                     |
| (d) Yes, $\lim_{x \rightarrow -1} f(x) = 4$ because $4 = \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$ |
- |                                                                                                                                                    |
|----------------------------------------------------------------------------------------------------------------------------------------------------|
| (a) No, $\lim_{x \rightarrow 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as $x$ approaches 0 |
| (b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 0 = 0$                                                                               |
| (c) $\lim_{x \rightarrow 0} f(x)$ does not exist because                                                                                           |

$\lim_{x \rightarrow 0^+}$ 

$f(x)$   
does  
not  
exist

6. (a) Yes,  $\lim_{x \rightarrow 0^+} g(x) = 0$  by the sandwich theorem since  $-\sqrt{x} \leq g(x) \leq \sqrt{x}$  when  $x > 0$   
 (b) No,  $\lim_{x \rightarrow 0^-} g(x)$  does not exist since  $\sqrt{x}$  is not defined for  $x < 0$   
 (c) Yes,  $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0^+} g(x) = 0$  since  $x = 0$  is a boundary point of the domain

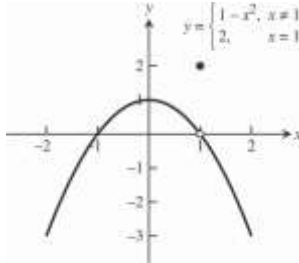
7. (a)



(b)  $\lim_{x \rightarrow 1^-} f(x) = 1 = \lim_{x \rightarrow 1^+} f(x)$

- (c) Yes,
- $\lim_{x \rightarrow 1} f(x) = 1$
- since the right-hand and left-hand limits exist and equal 1

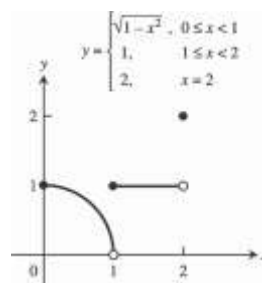
8. (a)



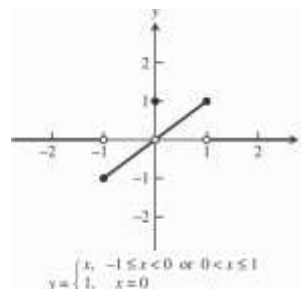
(b)  $\lim_{x \rightarrow 1^+} f(x) = 0 = \lim_{x \rightarrow 1^-} f(x)$

- (c) Yes,
- $\lim_{x \rightarrow 1} f(x) = 0$
- since the right-hand and left-hand limits exist and equal 0

9. (a) domain:  $0 \leq x \leq 2$   
 range:  $0 < y \leq 1$  and  $y = 2$   
 (b)  $\lim_{x \rightarrow c} f(x)$  exists for  $c$  belonging to  $(0, 1) \cup (1, 2)$   
 (c)  $x = 2$   
 (d)  $x = 0$



10. (a) domain:  $-\infty < x < \infty$   
 range:  $-1 \leq y \leq 1$   
 (b)  $\lim_{x \rightarrow c} f(x)$  exists for  $c$  belonging to  $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$   
 (c) none  
 (d) none



11.  $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5-1}} = \frac{3/2}{1/2} = \sqrt{3}$

12.  $\lim_{x \rightarrow 1^+} \frac{x-1}{x+2} = \frac{1-1}{1+2} = \frac{0}{3} = 0$

13.  $\lim_{x \rightarrow -2^+} \left( \frac{x}{x+1} \right) \left( \frac{2x+5}{x+x} \right) = \left( \frac{2}{2+1} \right) \left( \frac{-2}{-2+(-2)} \right) = \left( \frac{2}{3} \right) \left( \frac{-2}{-4} \right) = \left( \frac{2}{3} \right) \left( \frac{1}{2} \right) = \frac{1}{3}$

$$14. \lim_{x \rightarrow 1^-} \left( \frac{1}{x+1} \right) \left( \frac{x+6}{x} \right) \left( \frac{3-x}{7} \right) = \left( \frac{1}{1+1} \right) \left( \frac{1+6}{1} \right) \left( \frac{3-1}{7} \right) = \left( \frac{1}{2} \right) \left( \frac{7}{1} \right) \left( \frac{2}{7} \right) = 1$$

$$15. \lim_{h \rightarrow 0^+} \frac{h+4h+5-5}{\sqrt{h^2+4h+5}-5} = \lim_{h \rightarrow 0^+} \left( \frac{h+4h+5-5}{\sqrt{h^2+4h+5}-5} \right) \left( \frac{h+4h+5+5}{h+4h+5+5} \right) = \lim_{h \rightarrow 0^+} \frac{(h+4h+5)-5}{h(\sqrt{h^2+4h+5}+5)} \\ = \lim_{h \rightarrow 0^+} \frac{h(h+4)}{h(\sqrt{h^2+4h+5}+5)} = \frac{0+4}{\sqrt{5}+5} = \frac{2}{\sqrt{5}}$$

$$16. \lim_{h \rightarrow 0^-} \frac{\sqrt{6-5h+11h+6}}{\sqrt{h^2+4h+5}} = \lim_{h \rightarrow 0^-} \left( \frac{6-5h+11h+6}{h^2+4h+5} \right) \left( \frac{6+5h+11h+6}{6+5h+11h+6} \right) \\ = \lim_{h \rightarrow 0^-} \frac{6-(5h^2+11h+6)}{h(\sqrt{6+5h+11h+6})} = \lim_{h \rightarrow 0^-} \frac{-h(5h+11)}{h(\sqrt{6+5h+11h+6})} = \frac{-(0+11)}{6+\sqrt{6}} = -\frac{11}{6}$$

$$17. (a) \lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} = \lim_{x \rightarrow -2^+} (x+3) \frac{(x+2)}{(x+2)} \quad (|x+2| = (x+2) \text{ for } x > -2) \\ = \lim_{x \rightarrow -2^+} (x+3) = ((-2)+3) = 1$$

$$(b) \lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2} = \lim_{x \rightarrow -2^-} (x+3) \frac{-(x+2)}{(x+2)} \quad (|x+2| = -(x+2) \text{ for } x < -2) \\ = \lim_{x \rightarrow -2^-} (x+3)(-1) = -((-2)+3) = -1$$

$$18. (a) \lim_{x \rightarrow 1^+} \frac{\sqrt{2x(x-1)}}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{\sqrt{2x(x-1)}}{(x-1)} \quad (|x-1| = x-1 \text{ for } x > 1) \\ = \lim_{x \rightarrow 1^+} \sqrt{2x} = \sqrt{2}$$

$$(b) \lim_{x \rightarrow 1^-} \frac{\sqrt{2x(x-1)}}{|x-1|} = \lim_{x \rightarrow 1^-} \frac{\sqrt{2x(x-1)}}{-(x-1)} \quad (|x-1| = -(x-1) \text{ for } x < 1) \\ = \lim_{x \rightarrow 1^-} -\sqrt{2x} = -\sqrt{2}$$

$$19. (a) \text{ If } 0 < x < \frac{\pi}{2}, \text{ then } \sin x > 0, \text{ so that } \lim_{x \rightarrow 0^+} \frac{|\sin x|}{\sin x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{\sin x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$(b) \text{ If } -\frac{\pi}{2} < x < 0, \text{ then } \sin x < 0, \text{ so that } \lim_{x \rightarrow 0^-} \frac{|\sin x|}{\sin x} = \lim_{x \rightarrow 0^-} \frac{-\sin x}{\sin x} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$20. (a) \text{ If } 0 < x < \frac{\pi}{2}, \text{ then } \cos x < 1, \text{ so that } \lim_{x \rightarrow 0^+} \frac{1-\cos x}{|\cos x-1|} = \lim_{x \rightarrow 0^+} \frac{1-\cos x}{-(\cos x-1)} = \lim_{x \rightarrow 0^+} \frac{1-\cos x}{1-\cos x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$(b) \text{ If } -\frac{\pi}{2} < x < 0, \text{ then } \cos x < 1, \text{ so that } \lim_{x \rightarrow 0^-} \frac{\cos x-1}{|\cos x-1|} = \lim_{x \rightarrow 0^-} \frac{\cos x-1}{-(\cos x-1)} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$21. (a) \lim_{\theta \rightarrow 3^+} \frac{t-\theta}{\theta} = \frac{3}{3} = 1 \quad (b) \lim_{\theta \rightarrow 3^-} \frac{t-\theta}{\theta} = \frac{2}{3}$$

$$22. (a) \lim_{t \rightarrow 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0 \quad (b) \lim_{t \rightarrow 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$$

$$23. \lim_{x \rightarrow 0} \frac{\sin 2\theta}{\sin \theta} = \lim_{x \rightarrow 0} \frac{\sin x}{\sin x} = 1 \quad (\text{where } x = \sqrt{2}\theta)$$



$$\theta \rightarrow 0 \quad \sqrt{2\theta} \quad x \rightarrow 0 \quad x$$

$$24. \lim_{t \rightarrow 0} \frac{\sin kt}{t} = \lim_{t \rightarrow 0} \frac{k \sin t}{kt} = \lim_{\theta \rightarrow 0} \frac{k \sin \theta}{\theta} = k \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = k \cdot 1 = k \quad (\text{where } \theta = kt)$$

$$25. \lim_{y \rightarrow 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \rightarrow 0} \frac{3 \sin 3y}{3y} = \frac{3}{4} \lim_{y \rightarrow 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{3}{4} \quad (\text{where } \theta = 3y)$$

$$26. \lim_{h \rightarrow 0^-} \frac{\sin 3}{\frac{h}{h}} = \lim_{h \rightarrow 0^-} \left( \frac{1}{3} \cdot \frac{\sin 3}{1} \right) = \frac{1}{3} \lim_{h \rightarrow 0^-} \frac{\sin 3}{1} = \frac{1}{3} \cdot 1 = \frac{1}{3} \quad (\text{where } \theta = 3h)$$

$$27. \lim_{x \rightarrow 0} \frac{\tan 2x}{x} = \lim_{x \rightarrow 0} \frac{\left( \frac{\sin 2x}{\cos 2x} \right)}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{x \cos 2x} = \left( \lim_{x \rightarrow 0} \frac{1}{\cos 2x} \right) \left( \lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} \right) = 1 \cdot 2 = 2$$

$$28. \lim_{t \rightarrow 0} \frac{2t}{\tan t} = 2 \lim_{t \rightarrow 0} \frac{t}{\left( \frac{\sin t}{\cos t} \right)} = 2 \lim_{t \rightarrow 0} \frac{t \cos t}{\sin t} = 2 \left( \lim_{t \rightarrow 0} \cos t \right) \left( \lim_{t \rightarrow 0} \frac{t}{\sin t} \right) = 2 \cdot 1 \cdot 1 = 2$$

$$29. \lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \rightarrow 0} \left( \frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x} \right) = \left( \lim_{x \rightarrow 0} \frac{x}{\sin 2x} \right) \left( \lim_{x \rightarrow 0} \frac{1}{\cos 5x} \right) = \left( \frac{1}{2} \cdot 1 \right) (1) = \frac{1}{2}$$

$$30. \lim_{x \rightarrow 0} 6x^2 (\cot x)(\csc 2x) = \lim_{x \rightarrow 0} \frac{6x^2 \cos x}{\sin x \sin 2x} = \lim_{x \rightarrow 0} \left( 3 \cos x \cdot \frac{x}{\sin x} \cdot \frac{2x}{\sin 2x} \right) = 3 \cdot 1 \cdot 1 = 3$$

$$31. \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x} = \lim_{x \rightarrow 0} \left( \frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x} \right) = \lim_{x \rightarrow 0} \left( \frac{x}{\sin x} \cdot \frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \frac{x}{\sin x}$$

$$= \lim_{x \rightarrow 0} \left( \frac{1}{\sin x} \right) \cdot \lim_{x \rightarrow 0} \left( \frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \left( \frac{\sin x}{\sin x} \right) = (1)(1) + 1 = 2$$

$$32. \lim_{x \rightarrow 0} \frac{x - x + \sin x}{2x} = \lim_{x \rightarrow 0} \left( \frac{x}{2} - \frac{1}{2} + \frac{1}{2} \left( \frac{\sin x}{x} \right) \right) = 0 - \frac{1}{2} + \frac{1}{2} (1) = 0$$

$$33. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(2 \cos \theta)(1 + \cos \theta)} = \frac{0}{(2)(2)} = 0$$

$$34. \lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{\frac{x(1 - \cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{9x}}{\left( \frac{\sin 3x}{3x} \right)^2} = \frac{\frac{1}{9} \lim_{x \rightarrow 0} \left( \frac{1 - \cos x}{x} \right)}{\lim_{x \rightarrow 0} \left( \frac{\sin 3x}{3x} \right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$$

$$35. \lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = 1 - \cos t \rightarrow 0 \text{ as } t \rightarrow 0$$

$$36. \lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = \sin h \rightarrow 0 \text{ as } h \rightarrow 0$$

$$37. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \left( \frac{\sin \theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta} \right) = \frac{1}{2} \lim_{\theta \rightarrow 0} \left( \frac{\sin \theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta} \right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$38. \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x} = \lim_{x \rightarrow 0} \left( \frac{\sin 5x}{\sin 4x} \cdot \frac{4x}{5x} \cdot \frac{5}{4} \right) = \frac{5}{4} \lim_{x \rightarrow 0} \left( \frac{\sin 5x}{5x} \cdot \frac{4x}{\sin 4x} \right) = \frac{5}{4} \cdot 1 \cdot 1 = \frac{5}{4}$$

$$39. \lim_{\theta \rightarrow 0} \theta \cos \theta = 0 \cdot 1 = 0$$

$$40. \lim_{\theta \rightarrow 0} \sin \theta \cot 2\theta = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{2 \sin \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\cos 2\theta}{2 \cos \theta} = \frac{1}{2}$$

41.  $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x} = \lim_{x \rightarrow 0} \left( \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \right) = \lim_{x \rightarrow 0} \left( \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8} \right)$   
 $= \frac{3}{8} \lim_{x \rightarrow 0} \left( \frac{1}{\cos 3x} \right) \left( \frac{\sin 3x}{3x} \right) \left( \frac{8x}{\sin 8x} \right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$
42.  $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y} = \lim_{y \rightarrow 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim_{y \rightarrow 0} \left( \frac{\sin 3y}{y} \right) \left( \frac{\sin 4y}{\cos 4y} \right) \left( \frac{\cos 5y}{\sin 5y} \right) \left( \frac{3 \cdot 4 \cdot 5y}{3 \cdot 4 \cdot 5y} \right)$   
 $= \lim_{y \rightarrow 0} \left( \frac{\sin 3y}{3y} \right) \left( \frac{\sin 4y}{4y} \right) \left( \frac{5y}{\sin 5y} \right) \left( \frac{\cos 5y}{\cos 4y} \right) \left( \frac{3 \cdot 4}{5} \right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12}{5} = \frac{12}{5}$
43.  $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta}}{\theta^2 \frac{\cos 3\theta}{\sin 3\theta}} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta^2 \cos \theta \cos 3\theta} = \lim_{\theta \rightarrow 0} \left( \frac{\sin \theta}{\theta} \right) \left( \frac{1}{\theta \cos \theta} \right) \left( \frac{1}{\cos 3\theta} \right) = (1)(1) \left( \frac{1}{1 \cdot 1} \right) = 1$   
 $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta^2 \cos \theta \cos 3\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\theta \cos \theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos 3\theta} = 1 \cdot \lim_{\theta \rightarrow 0} \frac{1}{\theta \cos \theta} \cdot 1 = \lim_{\theta \rightarrow 0} \frac{1}{\theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\theta} \cdot 1 = \lim_{\theta \rightarrow 0} \frac{1}{\theta} = \infty$
44.  $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2 \theta \frac{\cos^2 2\theta}{\sin^2 2\theta}} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta}{\sin^2 \theta \cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta (2 \sin \theta \cos \theta)^2}{\sin^2 \theta \cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta (4 \sin^2 \theta \cos^2 \theta)}{\sin^2 \theta \cos^2 2\theta \sin 4\theta}$   
 $= \lim_{\theta \rightarrow 0} \frac{4\theta \cos 4\theta \cos^2 \theta}{\cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \left( \frac{4\theta}{\sin 4\theta} \right) \left( \frac{\cos 4\theta \cos^2 \theta}{\cos^2 2\theta} \right) = \lim_{\theta \rightarrow 0} \left( \frac{1}{\frac{\sin 4\theta}{4\theta}} \right) \left( \frac{\cos 4\theta \cos^2 \theta}{\cos^2 2\theta} \right) = \left( \frac{1}{1} \right) \left( \frac{1 \cdot 1^2}{1^2} \right) = 1$
45.  $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x} = \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x} \cdot \frac{1 + \cos 3x}{1 + \cos 3x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 3x}{2x(1 + \cos 3x)} = \lim_{x \rightarrow 0} \frac{\sin^2 3x}{2x(1 + \cos 3x)} = \lim_{x \rightarrow 0} \frac{3}{2} \cdot \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{1 + \cos 3x}$   
 $= \lim_{\theta \rightarrow 0} \frac{3}{2} \cdot \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} = \frac{3}{2} (1) \left( \frac{0}{1+1} \right) = 0 \quad (\text{where } \theta = 3x)$
46.  $\lim_{x \rightarrow 0} \frac{\cos^2 x - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x (\cos x - 1)}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x (\cos x - 1)}{x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x \rightarrow 0} \frac{\cos x (\cos^2 x - 1)}{x^2 (\cos x + 1)} = \lim_{x \rightarrow 0} \frac{\cos x (-\sin^2 x)}{x^2 (\cos x + 1)}$   
 $= \lim_{x \rightarrow 0} \left\{ -\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{\cos x}{\cos x + 1} \right\} = -(1)(1) \cdot \frac{1}{1+1} = -\frac{1}{2}$
47. Yes. If  $\lim_{x \rightarrow a^+} f(x) = L = \lim_{x \rightarrow a^-} f(x)$ , then  $\lim_{x \rightarrow a} f(x) = L$ . If  $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ , then  $\lim_{x \rightarrow a} f(x)$  does not exist.
48. Since  $\lim_{x \rightarrow c} f(x) = L$  if and only if  $\lim_{x \rightarrow c^+} f(x) = L$  and  $\lim_{x \rightarrow c^-} f(x) = L$ , then  $\lim_{x \rightarrow c} f(x)$  can be found by calculating  $\lim_{x \rightarrow c^+} f(x)$ .
49. If  $f$  is an odd function of  $x$ , then  $f(-x) = -f(x)$ . Given  $\lim_{x \rightarrow 0^+} f(x) = 3$ , then  $\lim_{x \rightarrow 0^-} f(x) = -3$ .
50. If  $f$  is an even function of  $x$ , then  $f(-x) = f(x)$ . Given  $\lim_{x \rightarrow 2^-} f(x) = 7$  then  $\lim_{x \rightarrow -2^+} f(x) = 7$ . However, nothing can be said about  $\lim_{x \rightarrow -2^-} f(x)$  because we don't know  $\lim_{x \rightarrow 2^+} f(x)$ .
51.  $I = (5, 5 + \delta) \Rightarrow 5 < x < 5 + \delta$ . Also,  $\sqrt{x-5} < \square \Rightarrow x-5 < \square^2 \Rightarrow x < 5 + \square^2$ . Choose  $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 5^+} \sqrt{x-5} = 0$ .
52.  $I = (4 - \delta, 4) \Rightarrow 4 - \delta < x < 4$ . Also,  $\sqrt{4-x} < \square \Rightarrow 4-x < \square^2 \Rightarrow x > 4 - \square^2$ . Choose  $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 4^-} \sqrt{4-x} = 0$ .

53. As  $x \rightarrow 0^-$  the number  $x$  is always negative. Thus,  $\left| \frac{x}{|x|} - (-1) \right| < \square \Rightarrow \left| \frac{x}{-x} + 1 \right| < \square \Rightarrow 0 < \square$  which is always true independent of the value of  $x$ . Hence we can choose any  $\delta > 0$  with  $-\delta < x < 0 \Rightarrow \lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$ .
54. Since  $x \rightarrow 2^+$  we have  $x > 2$  and  $|x-2| = x-2$ . Then,  $\left| \frac{x-2}{|x-2|} - 1 \right| = \left| \frac{x-2}{x-2} - 1 \right| < \square \Rightarrow 0 < \square$  which is always true so long as  $x > 2$ . Hence we can choose any  $\delta > 0$ , and thus  $2 < x < 2 + \delta \Rightarrow \left| \frac{x-2}{|x-2|} - 1 \right| < \square$ . Thus,  $\lim_{x \rightarrow 2^+} \frac{x-2}{|x-2|} = 1$ .
55. (a)  $\lim_{x \rightarrow 400^+} \lfloor x \rfloor = 400$ . Just observe that if  $400 < x < 401$ , then  $\lfloor x \rfloor = 400$ . Thus if we choose  $\delta = 1$ , we have for any number  $\square > 0$  that  $400 < x < 400 + \delta \Rightarrow \lfloor x \rfloor - 400 = |400 - 400| = 0 < \square$
- (b)  $\lim_{x \rightarrow 400^-} \lfloor x \rfloor = 399$ . Just observe that if  $399 < x < 400$  then  $\lfloor x \rfloor = 399$ . Thus if we choose  $\delta = 1$ , we have for any number  $\square > 0$  that  $400 - \delta < x < 400 \Rightarrow \lfloor x \rfloor - 399 = |399 - 399| = 0 < \square$
- (c) Since  $\lim_{x \rightarrow 400^+} \lfloor x \rfloor \neq \lim_{x \rightarrow 400^-} \lfloor x \rfloor$  we conclude that  $\lim_{x \rightarrow 400} \lfloor x \rfloor$  does not exist.
56. (a)  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{x} = \sqrt{0} = 0$ ;  $|\sqrt{x} - 0| < \square \Rightarrow -\square < \sqrt{x} < \square \Rightarrow 0 < x < \square^2$  for  $x$  positive. Choose  $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 0^+} f(x) = 0$ .
- (b)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$  by the sandwich theorem since  $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$  for all  $x \neq 0$ . Since  $|x^2 - 0| = |-x^2 - 0| = x^2 < \square$  whenever  $|x| < \sqrt{\square}$ , we choose  $\delta = \sqrt{\square}$  and obtain  $|x^2 \sin\left(\frac{1}{x}\right) - 0| < \square$  if  $-\delta < x < 0$ .
- (c) The function  $f$  has limit 0 at  $x_0 = 0$  since both the right-hand and left-hand limits exist and equal 0.

## 2.5 CONTINUITY

- No, discontinuous at  $x = 2$ , not defined at  $x = 2$
  - No, discontinuous at  $x = 3$ ,  $1 = \lim_{x \rightarrow 3^-} g(x) \neq g(3) = 1.5$
  - Continuous on  $[-1, 3]$
  - No, discontinuous at  $x = 1$ ,  $1.5 = \lim_{x \rightarrow 1^-} k(x) \neq \lim_{x \rightarrow 1^+} k(x) = 0$
- Yes
    - Yes
  - Yes,  $\lim_{x \rightarrow -1^+} f(x) = 0$
    - Yes
  - Yes,  $f(1) = 1$
    - No
  - Yes,  $\lim_{x \rightarrow 1} f(x) = 2$
    - No
  - No
    - No
  - $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$
  - $f(2) = 0$ , since  $\lim_{x \rightarrow 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \rightarrow 2^+} f(x)$

10.  $f(1)$  should be changed to  $2 = \lim_{x \rightarrow 1} f(x)$
11. Nonremovable discontinuity at  $x = 1$  because  $\lim_{x \rightarrow 1} f(x)$  fails to exist ( $\lim_{x \rightarrow 1^-} f(x) = 1$  and  $\lim_{x \rightarrow 1^+} f(x) = 0$ ).  
Removable discontinuity at  $x = 0$  by assigning the number  $\lim_{x \rightarrow 0} f(x) = 0$  to be the value of  $f(0)$  rather than  $f(0) = 1$ .
12. Nonremovable discontinuity at  $x = 1$  because  $\lim_{x \rightarrow 1} f(x)$  fails to exist ( $\lim_{x \rightarrow 1^-} f(x) = 2$  and  $\lim_{x \rightarrow 1^+} f(x) = 1$ ).  
Removable discontinuity at  $x = 2$  by assigning the number  $\lim_{x \rightarrow 2} f(x) = 1$  to be the value of  $f(2)$  rather than  $f(2) = 2$ .
13. Discontinuous only when  $x - 2 = 0 \Rightarrow x = 2$
14. Discontinuous only when  $(x + 2)^2 = 0 \Rightarrow x = -2$
15. Discontinuous only when  $x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0 \Rightarrow x = 3$  or  $x = 1$
16. Discontinuous only when  $x^2 - 3x - 10 = 0 \Rightarrow (x - 5)(x + 2) = 0 \Rightarrow x = 5$  or  $x = -2$
17. Continuous everywhere. ( $|x - 1| + \sin x$  defined for all  $x$ ; limits exist and are equal to function values.)
18. Continuous everywhere. ( $|x| + 1 \neq 0$  for all  $x$ ; limits exist and are equal to function values.)
19. Discontinuous only at  $x = 0$
20. Discontinuous at odd integer multiples of  $\frac{\pi}{2}$ , i.e.,  $x = (2n - 1)\frac{\pi}{2}$ ,  $n$  an integer, but continuous at all other  $x$ .
21. Discontinuous when  $2x$  is an integer multiple of  $\pi$ , i.e.,  $2x = n\pi$ ,  $n$  an integer  $\Rightarrow x = \frac{n\pi}{2}$ ,  $n$  an integer, but continuous at all other  $x$ .
22. Discontinuous when  $\frac{\pi x}{2}$  is an odd integer multiple of  $\frac{\pi}{2}$ , i.e.,  $\frac{\pi x}{2} = (2n - 1)\frac{\pi}{2}$ ,  $n$  an integer  $\Rightarrow x = 2n - 1$ ,  $n$  an integer (i.e.,  $x$  is an odd integer). Continuous everywhere else.
23. Discontinuous at odd integer multiples of  $\frac{\pi}{2}$ , i.e.,  $x = (2n - 1)\frac{\pi}{2}$ ,  $n$  an integer, but continuous at all other  $x$ .
24. Continuous everywhere since  $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 1 + \sin^2 x \geq 1$ ; limits exist and are equal to the function values.
25. Discontinuous when  $2x + 3 < 0$  or  $x < -\frac{3}{2} \Rightarrow$  continuous on the interval  $\left[-\frac{3}{2}, \infty\right)$ .
26. Discontinuous when  $3x - 1 < 0$  or  $x < \frac{1}{3} \Rightarrow$  continuous on the interval  $\left[\frac{1}{3}, \infty\right)$ .
27. Continuous everywhere:  $(2x - 1)^{1/3}$  is defined for all  $x$ ; limits exist and are equal to function values.
28. Continuous everywhere:  $(2 - x)^{1/5}$  is defined for all  $x$ ; limits exist and are equal to function values.
29. Continuous everywhere since  $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \rightarrow 3} (x + 2) = 5 = g(3)$

30. Discontinuous at  $x = -2$  since  $\lim_{x \rightarrow -2} f(x)$  does not exist while  $f(-2) = 4$ .
31. Discontinuous at  $x = 1$ ;  $\lim_{x \rightarrow 1^+} (x^2 + 2) = 3$ , but  $\lim_{x \rightarrow 1^-} e^x = e$ , so that  $\lim_{x \rightarrow 1} f(x)$  does not exist while  $f(1) = e$ ; and  $\lim_{x \rightarrow 0^-} (1 - x) = 1 = \lim_{x \rightarrow 0^+} e^x$ , so that  $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
32. Discontinuous at  $x = \ln 2$ , since  $2 - e^x = 0 \Rightarrow e^x = 2 \Rightarrow \ln e^x = \ln 2 \Rightarrow x = \ln 2$
33.  $\lim_{x \rightarrow \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$ , and function continuous at  $x = \pi$
34.  $\lim_{t \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right) = \sin\left(\frac{\pi}{2} \cos(\tan(0))\right) = \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$ , and function continuous at  $t = 0$ .
35.  $\lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1) = \lim_{y \rightarrow 1} \sec(y \sec^2 y - \sec^2 y) = \lim_{y \rightarrow 1} \sec((y - 1) \sec^2 y) = \sec((1 - 1) \sec^2 1) = \sec 0 = 1$ , and function continuous at  $y = 1$ .
36.  $\lim_{x \rightarrow 0} \tan\left[\frac{\pi}{4} \cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4} \cos(\sin(0))\right] = \tan\left(\frac{\pi}{4} \cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$ , and function continuous at  $x = 0$ .
37.  $\lim_{t \rightarrow 0} \cos\left[\frac{\pi}{\sqrt{19-3 \sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19-3 \sec 0}}\right] = \cos \frac{\pi}{\sqrt{16}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ , and function continuous at  $t = 0$ .
38.  $\lim_{x \rightarrow \frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3} \tan x} = \sqrt{\csc^2\left(\frac{\pi}{6}\right) + 5\sqrt{3} \tan\left(\frac{\pi}{6}\right)} = \sqrt{4 + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{9} = 3$ , and function continuous at  $x = \frac{\pi}{6}$ .
39.  $\lim_{x \rightarrow 0^+} \sin\left(\frac{\pi}{2} e^{\sqrt{x}}\right) = \sin\left(\frac{\pi}{2} e^0\right) = \sin\left(\frac{\pi}{2}\right) = 1$ , and the function is continuous at  $x = 0$ .
40.  $\lim_{x \rightarrow 1} \cos^{-1}(\ln \sqrt{x}) = \cos^{-1}(\ln \sqrt{1}) = \cos^{-1}(0) = \frac{\pi}{2}$ , and the function is continuous at  $x = 1$ .
41.  $g(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = x+3, x \neq 3 \Rightarrow g(3) = \lim_{x \rightarrow 3} (x+3) = 6$
42.  $h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t+5, t \neq 2 \Rightarrow h(2) = \lim_{t \rightarrow 2} (t+5) = 7$
43.  $f(s) = \frac{s^3-1}{s^3-1} = \frac{(s^2+s+1)(s-1)}{(s+1)(s-1)} = \frac{s^2+s+1}{s+1}, s \neq 1 \Rightarrow f(1) = \lim_{s \rightarrow 1} \left(\frac{s^2+s+1}{s+1}\right) = \frac{3}{2}$
44.  $g(x) = \frac{x^2-16}{x^2-3x-4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1}, x \neq -4 \Rightarrow g(4) = \lim_{x \rightarrow 4} \left(\frac{x+4}{x+1}\right) = \frac{8}{5}$
45. As defined,  $\lim_{x \rightarrow 3^-} f(x) = (3)^2 - 1 = 8$  and  $\lim_{x \rightarrow 3^+} (2a)(3) = 6a$ . For  $f(x)$  to be continuous we must have  $6a = 8 \Rightarrow a = \frac{4}{3}$ .